Successive and Two-Stage Systematic Encoding of Polar Subcodes

Ruslan Morozov, Peter Trifonov
{rmorozov, petert}@dcn.icc.spbstu.ru

***Abstract*—Fast methods for systematic encoding of polar subcodes are proposed. Polar subcodes with systematic encoding are shown to provide lower bit error rate compared to the case of non-systematic encoding as well as compared to the case of systematic polar codes. The proposed algorithms can be used for 5G polar subcodes, allowing one to reduce BER, decoding complexity and latency.**

I. Introduction

Polar codes [1] is a novel class of capacity achieving codes with low construction, encoding and decoding complexity. However, their finite-length frame error probability (FER) is quite poor. It is both due to suboptimality of the successive cancellation decoding algorithm and low minimum distance. Polar codes with CRC under Tal-Vardy list decoding algorithm [2] provide performance close to that of LDPC codes, while polar subcodes, also known as parity-check concatenated polar codes, were shown to provide even better performance [3–5] due to reduced error coefficient or increased minimum distance. The latter type of codes was accepted to the 5G standard [6].

Observe that the Tal-Vardy algorithm constructs codewords of a polar code, but not the corresponding information vectors. Recovering the information vectors from the list decoder would require either storing additional arrays, which would lead to $O(n^2)$ copy operations during decoding, or multiplying the obtained codewords by the Arikan matrix, which costs $\frac{1}{2}n\log_2 n$ operations, and at least $\log_2 n$ clockticks.

Systematic encoding is a method of mapping data bits to a code, so that the data bits appear at some fixed positions in a codeword. Systematic encoding allows one to directly access data bits from a codeword, decreasing thus the decoding complexity and latency. Furthermore, systematic encoding of component codes may be helpful in the case of concatenated coding schemes.

Many encoding algorithms were proposed for systematic polar codes [7–11]. However, to the best of our knowledge, no study on systematic encoding of polar subcodes has been done. In this paper, systematic encoders [10] and [9] for polar codes are generalized to the case of polar subcodes. The generalized systematic encoders are called two-stage and successive, respectively. We show the sufficient condition of applicability of the two-stage encoder, and show how the dynamic freezing constraints of a polar subcode need to be modified to allow the successive encoding.

The authors are with St. Petersburg Polytechnic University, Russia.

II. Background

A. Systematic encoding of polar codes

A polar code of length $n = 2^m$ is a set of vectors $c_0^{n-1} = u_0^{n-1}A$, where $A = \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}^{\otimes m}$ is the polarizing transformation (PT), by $M^{\otimes t}$ we denote the t-th Kronecker power of matrix M, and $u_i = 0, i \in \phi$, $\phi \subset [n]$. By $[n]$ we denote set $[n] = \{0, \dots, n-1\}$. The dimension of the code is $n - |\phi|$. We call ϕ *the set of frozen symbols*, and $\alpha = [n] \setminus \phi$ *the set of information symbols*.

The standard non-systematic encoding algorithm is as follows. Place the data bits a_0^{k-1} in vector u_0^{n-1} on positions from α, filling other positions of u_0^{n-1} with zeroes, i.e. set $u_\alpha = a, u_\phi = 0$. Then the codeword can be obtained as $c = u_0^{n-1}A = aE_{\alpha,[n]}A = aA_{\alpha,[n]}$, where E is the $n \times n$ identity matrix. Throughout the paper by $M_{I,J}$ we denote the submatrix of M which consists of rows and columns with indices from sets I and J, respectively. The similar notation M_I is used if M is a vector.

Let $\mathcal{C} \subset \mathbb{F}_2^n$ be a binary linear block code of dimension k, where $\mathbb{F}_2$ is the binary field. Then a bijective function $f : \mathbb{F}_2^k \to \mathcal{C}$ for which there exists a set $\zeta \subset [n], |\zeta| = k$ such that $\forall a_0^{k-1} \in \mathbb{F}_2^k : f(a_0^{k-1})_\zeta = a_0^{k-1}$ is called a *systematic encoder*. Such ζ is called a *systematic set*. The problem of systematic encoding can be formulated as follows: given a data vector of length k and set ζ, obtain vector c of length n, such that:

$$\begin{cases} c_\zeta = a_0^{k-1} \\ c_0^{n-1} \in \mathcal{C} \end{cases} \tag{1}$$

Many systematic encoding algorithms were suggested for classical polar codes. We consider here two encoders with $\zeta = \alpha$: the low-latency Sarkis-Tal-Giard-Vardy-Thibeault-Gross (STGVTG) encoder [10] and low-complexity Vangala-Hong-Viterbo (VHV) encoder [9]. Observe also that the STGVTG encoder imposes some restrictions on the set of information symbols α, while the VHV encoder is applicable for any polar code. We propose generalizations of these algorithms to the case of polar subcodes.

B. The STGVTG encoder

In [10] a low-latency method for systematic encoding of polar codes is introduced. It consists of the following steps:

1) Encode a_0^{k-1} with the classical encoding algorithm: $v_0^{n-1} = a_0^{k-1}A_{\alpha,[n]}$.

2) Set elements of v with indices from ϕ to zero: $v_\phi \leftarrow 0$.
3) Compute the codeword as $c_0^{n-1} = v_0^{n-1} A$.

This method can be represented in matrix form as

$$c_0^{n-1} = a_0^{k-1} A_{\alpha,[n]} E_{[n],\alpha} E_{\alpha,[n]} A = a_0^{k-1} A_{\alpha,\alpha} A_{\alpha,[n]}. \quad (2)$$

Observe that if $A_{\alpha,\alpha} A_{\alpha,\alpha} = E_{[k],[k]}$, then (2) corresponds to systematic encoding with $\zeta = \alpha$. It is shown in [10] that $A^2_{\alpha,\alpha}$ is identity matrix if the set of information symbols α satisfies the *domination contiguity* property (D.C.), i.e.

$$i, j \in \alpha \wedge i \preceq t \preceq j \implies t \in \alpha, \quad (3)$$

where for $i \in [2^m]$ by i_t we denote the t-th most significant bit in the binary expansion $i = \sum_{j=0}^{m-1} i_j 2^{m-1-j}$, and by $i \preceq j$ we mean $\forall t : i_t \leq j_t$.

The complexity of such encoding method is $\mathcal{O}_{STGVTG} = n \log_2 n$ operations, which corresponds to the complexity of two multiplications by A. It is easy to show that the latency is $\mathcal{L}_{STGVTG} = 2\log_2 n + 2$, provided that there are infinitely many processors operating in parallel.

C. The VHV encoder

Three effective systematic encoders were proposed in [9]. Here we consider only the algorithm *EncoderA* due to its non-recursive structure, low complexity and because it is easy to extend to the case of polar subcodes. We refer to this algorithm as the VHV encoder.

Consider factor graph corresponding to the PT. In the VHV encoding algorithm, frozen symbols are propagated from the leftmost layer to the rightmost, and information symbols are propagated from the rightmost layer to the leftmost. The codeword consists of elements from the rightmost layer of the factor graph. The complexity of the algorithm (not including assignments) equals $\mathcal{O}_{VHV} = \frac{1}{2} n \log_2 n$, which corresponds to multiplication by A. The latency heavily depends on implementation. It can be upper-bounded by $\mathcal{L}_{VHV} = O(n \log_2 n)$. In [11] a memory-efficient implementation of the VHV algorithm was proposed, which requires $O(n)$ memory.

D. Polar subcodes

A polar subcode [3] of length $n = 2^m$ is a set of vectors $c_0^{n-1} = u_0^{n-1} A$, where elements of the vector u_0^{n-1} satisfy

$$\begin{cases} u_i = 0, & i \in \phi \\ u_i = \sum_{j=0}^{i-1} V_{i,j} u_j, & i \in \delta \\ \delta \cap \phi = \emptyset \end{cases} \quad (4)$$

We assume that V is $n \times n$ matrix with $V_{i,j} = 0$ if $i \notin \delta$ or $j \geq i$. The set ϕ is called the set of *static* frozen symbols, and the set δ is called the set of *dynamic* frozen symbols. The set of information symbols is $\alpha = [n] \setminus (\phi \cup \delta)$. The dimension of the code is $k = |\alpha|$. Denote $\mathcal{D}_i$ the set of indices of non-zero entries of the i-th row of V. We consider here the most general case of binary polar subcodes, letting the structure of $\mathcal{D}_i$ be arbitrary. Note that classical polar codes can be considered as a special case of polar subcodes with $\delta = \emptyset$.

Let $\sigma = \alpha \cup \delta = \{\sigma_i\}$ be sorted in ascending order and let $|\sigma| = d$. Let us further define $D = V_{\alpha,\sigma}$ as a $k \times d$ matrix, which maps the information vector a_0^{k-1} to a vector containing both information and dynamic frozen symbols. Then non-systematic encoding can be implemented as $c_0^{n-1} = a_0^{k-1} D A_{\sigma,[n]}$.

III. Systematic Encoders for Polar Subcodes

In this section, we propose two generalizations of the systematic encoders of classical polar codes, described in Sections II-B and II-C, to the case of polar subcodes. They are described in sections III-A and III-B, respectively.

A. The Two-Stage Encoder

We propose the following two-stage encoder, which is a generalization of the STGVTG encoder to the case of polar subcodes.

1) Set $u_\alpha = a_0^{k-1}$ and $u_{\phi\cup\delta} = 0$. That is, the values of all dynamic frozen symbols are temporarily set to zero.
2) Compute $v_0^{n-1} = u_0^{n-1} A$.
3) Recalculate all frozen symbols $v_i, i \in \phi \cup \sigma$, according to (4), leaving the values of information symbols unchanged. Let w_0^{n-1} be the obtained vector.
4) Compute the codeword $c_0^{n-1} = w_0^{n-1} A$.

In matrix form this procedure can be written as

$$c_0^{n-1} = a_0^{k-1} A_{\alpha,\alpha} D A_{\sigma,[n]}. \quad (5)$$

Note that (5) defines a systematic encoder with information set $\zeta = \alpha$, iff

$$A_{\alpha,\alpha} D A_{\sigma,\alpha} = E_{[k],[k]}. \quad (6)$$

Theorem 1. *If α satisfies the domination contiguity property* (3) *and $(n-1) \in \alpha$, then* (6) *is true and* (5) *is a systematic encoder with the systematic set $\zeta = \alpha$.*

Proof. This proof is a generalization of the proof of the Theorem 1 from [10].

Assuming $(n-1) \in \alpha$, equation (3) leads to $i \in \alpha, j \succeq i \implies j \in \alpha$. For a given statement S, let us denote $\mathbb{I}[S] = 1$ if the statement is true and $\mathbb{I}[S] = 0$ otherwise.

Note that $A_{i,j} = \mathbb{I}[i \succeq j]$ and $D_{i,j} = \mathbb{I}[\alpha_i = \sigma_j \vee \alpha_i \in \mathcal{D}_{\sigma_j}]$. Denote $\mathcal{J} = \{t : \sigma_t \in \alpha\}$. Let $\mathcal{J} = \{\mathcal{J}_i\}_{i=0}^{k-1}$ be sorted in ascending order. Note that $t \in \mathcal{J} \wedge D_{i,t} = 1 \iff \alpha_i = \sigma_t$.

$$(DA_{\sigma,\alpha})_{i,j} = \sum_{t=0}^{d-1} D_{i,t} \cdot \mathbb{I}[\sigma_t \succeq \alpha_j] = \sum_{t \in \mathcal{J}} D_{i,t} \cdot \mathbb{I}[\sigma_t \succeq \alpha_j]$$
$$+ \sum_{t \notin \mathcal{J}} D_{i,t} \cdot \mathbb{I}[\sigma_t \succ \alpha_j] = \mathbb{I}[\alpha_i \succeq \alpha_j] + \sum_{t:\alpha_i \in \mathcal{D}_{\delta_t}} \underbrace{\mathbb{I}[\delta_t \succ \alpha_j]}_{0 \text{ by D.C.}}.$$

Here $\sum$ denotes summation in $\mathbb{F}_2$. Hence, one obtains

$$(A_{\alpha,\alpha} D A_{\sigma,\alpha})_{i,j} = \sum_{s=0}^{k-1} \mathbb{I}[\alpha_i \succeq \alpha_s] \cdot \mathbb{I}[\alpha_s \succeq \alpha_j]$$
$$= |\{s : \alpha_i \succeq \alpha_s \succeq \alpha_j\}| \bmod 2.$$

The last value is 1 only if $\alpha_i \succeq \alpha_j$, since $\succeq$ is a transitive relation. So, $\operatorname{supp} \alpha_i \supseteq \operatorname{supp} \alpha_j$. Due to D.C. $\alpha_i \succeq r \succeq \alpha_j \implies r \in \alpha$. But $\alpha_i \succeq r \succeq \alpha_j$ means $\operatorname{supp} \alpha_i \supseteq \operatorname{supp} r \supseteq \operatorname{supp} \alpha_j$. There are $2^{|\operatorname{supp} \alpha_i \setminus \operatorname{supp} \alpha_j|}$

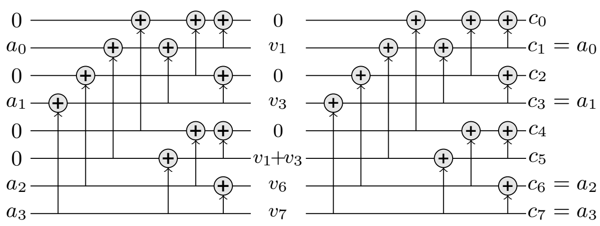

Fig. 1. The two-stage systematic encoder for a polar subcode

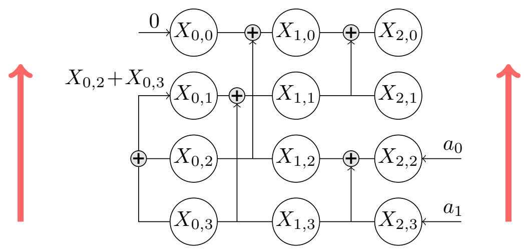

Fig. 2. The successive systematic encoder for a polar subcode

such values of r. Therefore, the last value is 1 if and only if $\operatorname{supp}\alpha_i \subseteq \operatorname{supp}\alpha_j$. So, $\operatorname{supp}\alpha_i = \operatorname{supp}\alpha_j$, i.e. $i = j$, which leads to $A_{\alpha,\alpha}DA_{\sigma,\alpha}$ being the identity matrix. □

An example of encoding for the case of polar subcode with $n = 8, k = 4, \phi = \{0,2,4\}, \delta = \{5\}, u_5 = u_1 + u_3$, is presented in Fig. 1.

The complexity of this generalization is $\mathcal{O}_{2S} = n\log_2 n + \sum_i |\mathcal{D}_i|$ operations. The first term corresponds to the complexity of two multiplications by A, and the second one is the complexity of evaluation of dynamic frozen symbols. The latency is $\mathcal{L}_{2S} = 2\log_2 n + \max_i |\mathcal{D}_i| + 2$, where the first term is the latency of two multiplications by A, the second term is the latency of computing dynamic freezing constraints, and the last one is the latency of initializing the input vector u_0^{n-1} and computing w_0^{n-1} from v_0^{n-1}.

B. The Successive Encoder

Observe that (4) assumes that each dynamic frozen symbol u_i depends on previous symbols $u_j, j < i$. However, the VHV encoding algorithm, described in subsection II-C, examines symbols of u_0^{n-1} in descending order of their indices, and one can use for computing u_i only the succeeding symbols u_j, $j > i$. We propose the generalization of VHV encoding to polar subcodes via the following preprocessing of dynamic freezing constraints.

Let α, ϕ, $\delta \subset [n]$, be the sets of information, static frozen and dynamic frozen symbols, respectively, and $\mathcal{D}_i$ be the set of terms in dynamic freezing constraints (4) for $u_i, i \in \delta$. Assume that $\beta_i = \min \mathcal{D}_i$ are distinct for $i \in \delta$. This condition can be enforced by performing elementary row operations with matrix V, which do not change the code. Let $\mathcal{D}'_{\beta_i} = \mathcal{D}_i \cup \{i\} \setminus \{\beta_i\}$. One can see that the modified polar subcode with sets $\delta' = \{\beta_i | i \in \delta\}$, $\alpha' = \alpha \cup \delta \setminus \delta'$, and sets of terms $\mathcal{D}'_i$, $i \in \delta'$, is equivalent to the initial polar subcode, but each symbol u_i now depends only on $u_j, j > i$.

The successive encoding consists of n propagation phases $\Phi = n-1 \dots 0$ on factorgraph of the PT, as shown in Fig. 2. Denote the vertices of the PT as $X_{i,j}$, where i and j correspond to the layer of the PT and the symbol index, respectively.

Consider encoding phase Φ. If $\Phi \notin \alpha'$, then $X_{0,\Phi}$ is assigned to the frozen value of u_Φ. The nodes $X_{\Phi,i}, i = 1 \dots m$ are updated one by one, using already computed values $X_{j,i-1}, j \geq \Phi$. Thus, the input vector of the PT satisfies freezing constraints, so that the first condition of (1) holds.

Let $\alpha' = \{\alpha'_i\}$ be sorted in ascending order. If $\Phi = \alpha'_l$, the information symbol $u_\Phi = a_l$ is placed in $X_{m,\Phi}$. The nodes $X_{\Phi,i}, i = m-1 \dots 0$ are updated, using known values of $X_{j,i+1}, j \geq \Phi$. The output codeword is $X_{m,i}, i \in [n]$, and since $X_{m,\alpha'_l} = a_l$, the second condition of (1) is satisfied. The systematic set is α', which may differ from the initial set α of information symbols.

Algorithm 1: Successive encoding

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Input: m, α, δ, (D_i)_{i∈δ}, a_0^{k-1}
obtain α', δ' and (D'_i)_{i∈δ'} as described above
n ← 2^m; X_{0,[n]\α'} ← 0; X_{m,α} ← a_0^{k-1}
for i ← n−1 ... 0 do
    if i ∈ α' then
        for λ ← m−1 ... 0 do
            if i_λ = 0 then
                X_{λ,i} ← X_{λ+1,i} ⊕ X_{λ+1,i+2^{m-1-λ}}
            else X_{λ,i} ← X_{λ+1,i};
    else
        if i ∈ δ' then X_{0,i} ← Σ_{j∈D'_i} X_{0,j};
        for λ ← 0 ... m−1 do
            if i_λ = 0 then
                X_{λ+1,i} ← X_{λ,i} ⊕ X_{λ,i+2^{m-1-λ}}
            else X_{λ+1,i} ← X_{λ,i};
return: X_{m,[n]}
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An example of calculations is presented in Fig. 2 for $m = 2$, $k = 2$. The initial polar subcode corresponds to $\phi = \{0\}$, $\alpha = \{1,2\}$, $\delta = \{3\}$, $\mathcal{D}_3 = \{1,2\}$, $u_3 = u_1 + u_2$. The modified sets are $\alpha' = \{2,3\}$, $\delta' = \{1\}$, $\mathcal{D}'_1 = \{2,3\}$, which define the same code. The generalized version of the VHV algorithm is presented as Algorithm 1. The complexity of the algorithm is $\mathcal{O}_{SS} = \frac{1}{2}n\log_2 n + \sum_i |\mathcal{D}'_i|$. The first term reflects the cost of the calculation flow derived from the VHV algorithm, while the second term corresponds to the complexity of evaluation of dynamic frozen symbols.

IV. Comparison of the Methods

Observe that the Tal-Vardy decoding algorithm produces a codeword $c_0^{n-1} = u_0^{n-1}A$, and not the information vector u_0^{n-1}. Systematic encoding enables one to obtain information bits directly from c_0^{n-1}, by avoiding the need to compute $u_0^{n-1} = c_0^{n-1}A$, which has complexity $\frac{1}{2}n\log_2 n$ and latency $\log_2 n$. The price for this is increased encoding complexity, which is typically not a critical issue. In Table I complexity and latency of systematic encoding algorithms for polar codes and

TABLE I
COMPLEXITY AND LATENCY COMPARISON OF PRESENTED ENCODING ALGORITHMS

Algorithm	Complexity	Latency
STGVTG [10]	$n\log_2 n$	$2\log_2 n + 2$
Two-stage	$n\log_2 n + \sum_i \lvert\mathcal{D}'_i\rvert$	$2\log_2 n + \max_i \lvert\mathcal{D}'_i\rvert + 2$
VHV [9]	$n/2 \cdot \log_2 n$	$O(n\log_2 n)$
Successive	$n/2 \cdot \log_2 n + \sum_i \lvert\mathcal{D}_i\rvert$	$O(n\log_2 n)$

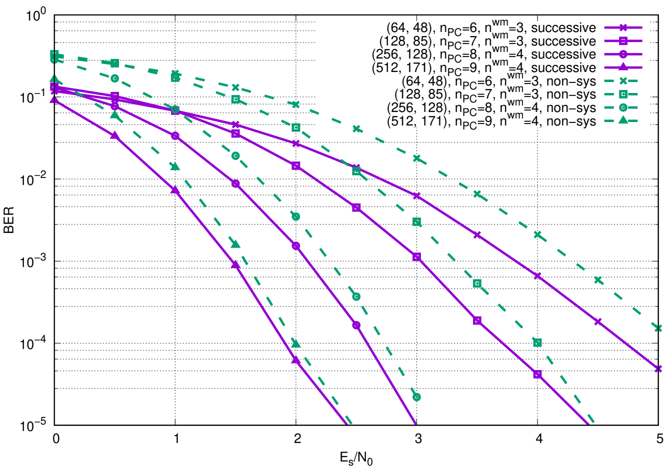

Fig. 3. Systematic versus non-systematic 5G polar subcodes [6]

polar subcodes are compared. The complexity gap between the original algorithms and their generalizations to the case of polar subcodes seems to be near-optimal, since it equals to the complexity of computing dynamic frozen symbols.

Fig. 3 presents simulation results illustrating BER of non-systematic encoding and successive systematic encoding of 5G polar subcodes with $n_{PC} = \log_2 n$ and $n_{PC}^{wm} = \lfloor n_{PC}/2 \rfloor$ (see [6] for the exact codes specification) under SCL [2] decoding with $L = 8$. Since systematic encoding does not change the set of codewords, frame error rate is the same for all encoding methods. However, the bit error rate (BER) is reduced, since close information vectors are mapped to close codewords. Systematic encoding of polar subcodes delivers up to 0.5 dB performance gain in terms of BER.

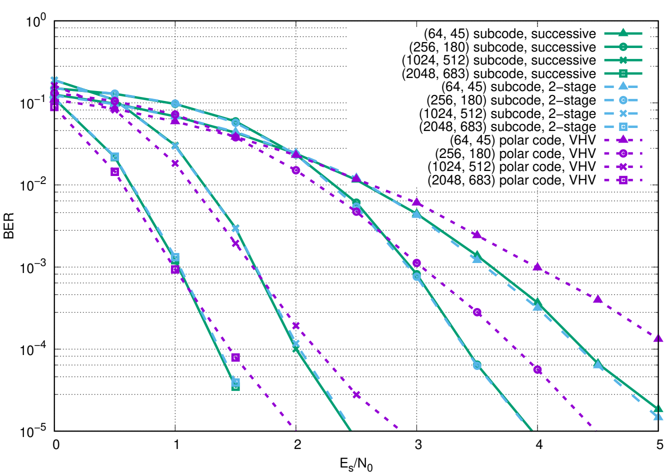

Fig. 4. Systematic polar subcodes [4] versus systematic Arikan polar codes.

Fig. 4 presents the BER for systematic polar codes and subcodes. The design SNR was optimized for all codes for target FER 10^{-3}. One can see that systematic polar subcodes provide lower BER compared to systematic polar codes. Observe that although two proposed algorithms in general do not lead to the same mapping of data vector to the code, the obtained BER is very close in both cases.

CONCLUSIONS

In this paper two efficient methods for systematic encoding for polar subcodes are presented. The proposed algorithms allow one to simplify decoding of polar subcodes by obtaining payload data directly from the restored codeword. Furthermore, we show that polar subcodes, similarly to polar codes, provide reduced BER under systematic encoding. The proposed approach applies also to the polar subcodes defined in 5G standard.

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