Star Polar Subcodes

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***Abstract*—A generalization of polar subcodes is proposed together with the associated list decoding algorithm. The proposed algorithm admits parallel implementation, provides lower latency and better performance compared to the Tal-Vardy list decoding method, with a slight increase in decoding complexity.**

I. Introduction

Polar codes is a novel class of error-correcting codes, which asymptotically achieve the symmetric capacity of memoryless channels, have low complexity construction, encoding and decoding algorithms [1]. However, their practical implementation faces several challenges.

First, minimum distance of moderate length polar codes scales as $O(\sqrt{n})$, where n is code length [2], which is too low for practical applications. This problem was addressed in [3], [4], where it was suggested to set some frozen symbols of the polarizing transformation not to 0, as in classical polar codes, but to some linear combinations of other symbols. The structure of these linear combinations can be derived either from a CRC error detecting code, or a check matrix of an extended BCH code.

Second, the performance of the successive cancellation decoding algorithm is far from that of a maximum likelihood decoder for polar codes. List [5] and sequential [6] decoding algorithms provide substantially better performance than the SC algorithm at the expense of higher complexity.

However, the latency of the SC algorithm and its list/sequential generalizations is too high for next generation wireless systems. This is due to strong data dependencies induced by the factor graph of the polar code, which prevent parallel implementation of the decoder.

In this paper we present a construction of star polar subcodes, which enables one to perform parallel decoding, and in some cases obtain slight performance gain with respect to conventional polar subcodes. A codeword of a star polar subcode essentially consists of a number of codewords of short polar codes, such that their information vectors have some linear dependencies. Decoding of the proposed codes is performed by running parallel instances of the Tal-Vardy list decoder for short polar codes, which are "synchronized", so that their output codewords constitute a valid codeword of the considered code.

The paper is organized as follows. Section II provides a background on polar codes and their decoding. The proposed code construction is introduced in Section III. A generalization of the Tal-Vardy list decoding algorithm to the case of star polar subcodes is presented in Section IV. Simulation results illustrating their performance are provided in Section V. Finally, some conclusions are drawn.

II. Background

A. Polar subcodes

A $(n=2^m,k,d)$ polar code over $\mathbb{F}_2$ is a set of vectors $c_0^{n-1}=u_0^{n-1}A_m$, where $a_i^j=(a_i,\dots,a_j)$, $A_m=\begin{pmatrix}1&0\\1&1\end{pmatrix}^{\otimes m}B_m$ is a matrix of the polarizing transformation, B_m is the bit-reversal permutation matrix, $F^{\otimes m}$ denotes m-times Kronecker product of matrix F with itself, $u_i=0, i\in\mathcal{F}$, and $\mathcal{F}\subset\{0,\dots,n-1\}$ is a set of $n-k$ frozen symbol indices. It is possible to show that the polarizing transformation together with a memoryless channel $\mathbf{W}(y|c)$ gives rise to n synthetic bit subchannels with transition probability functions

$$\mathbf{W}_m^{(i)}(y_0^{n-1},u_0^{i-1}|u_i)=\frac{1}{2^{n-1}}\sum_{u_{i+1}^{n-1}}\prod_{j=0}^{n-1}\mathbf{W}(y_j|(u_0^{n-1}A_m)_j).$$

One can compute the capacity $C_{m,i}$ and bit error rate $P_{m,i}$ in each of the subchannels $\mathbf{W}_m^{(i)}$ using the techniques presented in [7], [8]. The set $\mathcal{F}$ is typically constructed as the set of $n-k$ integers with the highest $P_{m,i}$ or lowest $C_{m,i}$.

It is possible to show that the minimum distance of classical Arikan polar codes is $O(\sqrt{n})$, which is too low for practical applications [2]. It was suggested in [4] to set frozen symbols $u_i, i\in\mathcal{F}$, not to zero, but to some linear combinations of $u_j, j<i$, i.e.

$$u_i=\sum_{j<i}V_{s_i,j}u_j, i\in\mathcal{F}, \tag{1}$$

where V is a $(n-k)\times n$ constraint matrix, such that distinct rows end[1] in distinct columns, and s_i is the index of the row ending in column i. The symbols u_i with non-zero r.h.s. in (1) are referred to as dynamic frozen.

Matrix V can be constructed so that codewords c_0^{n-1} belong to some (n,k',d) parent code with check matrix H and sufficiently high minimum distance d, and the SC decoding error probability $P(\mathcal{F})$ is minimized. This can be obtained by computing $V'=HA_m^T$, performing elementary row operations in order to ensure that all rows end in distinct columns, and appending additional $k'-k$ weight-1 rows, having 1's in positions with the highest $P_{m,i}$, such that no other row ends in these positions. The obtained codes are referred to as polar subcodes. Extended primitive narrow-sense BCH (e-BCH) codes were shown to be good parent codes. Polar subcodes of e-BCH codes were shown to provide substantially better performance compared to classical polar codes.

[1]Given some binary vector a_0^{n-1}, we say that it starts in position i and ends in position j iff $a_i=a_j=1$ and $a_s=a_t=0, 0\le s<i, j<t<n$.

B. List decoding

The successive cancellation (SC) decoding algorithm for polar subcodes makes decisions

$$\hat{u}_i = \begin{cases} \arg\max_{u_i \in \mathbb{F}_2} \mathbf{W}_m^{(i)}(y_0^{n-1}, \hat{u}_0^{i-1}|u_i), & i \notin \mathcal{F} \\ \sum_{j<i} V_{s_i,j} u_j, & i \in \mathcal{F}. \end{cases} \quad (2)$$

It is possible to implement these calculations with complexity $O(n \log n)$.

However, the estimates $\hat{u}_i$ obtained by the SC algorithm do not take into account freezing constraints on symbols $u_j, j > i$. This causes the SC algorithm to be highly suboptimal. This problem was addressed in [3], where it was suggested to keep at each phase i L most probable estimate vectors $\hat{u}_0^i$. Here we present a derivation of the min-sum [9] version of the Tal-Vardy algorithm, which will be used in the decoding algorithm for the proposed codes.

Let

$$\mathcal{W}_m^{(i)}(u_0^i|y_0^{n-1}) = \max_{u_{i+1}^{n-1} \in \mathbb{F}_2^{n-i-1}} \mathbf{W}_m^{(n-1)}(u_0^{n-1}|y_0^{n-1})$$

be the probability of the most likely continuation of path u_0^i in the code tree, without taking into account freezing constraints on symbols $u_j, j > i$. It can be seen that for $\lambda > 0$

$$\mathcal{W}_\lambda^{(2i)}(u_0^{2i}|y_0^{n-1}) = \max_{u_{2i+1} \in \mathbb{F}_2} \mathcal{W}_{\lambda-1}^{(i)}(u_{0,e}^{2i+1} \oplus u_{0,o}^{2i+1}|y_0^{\frac{n}{2}-1}) \cdot \mathcal{W}_{\lambda-1}^{(i)}(u_{0,o}^{2i+1}|y_{\frac{n}{2}}^{n-1}), \quad (3)$$

$$\mathcal{W}_\lambda^{(2i+1)}(u_0^{2i+1}|y_0^{n-1}) = \mathcal{W}_{\lambda-1}^{(i)}(u_{0,e}^{2i+1} \oplus u_{0,o}^{2i+1}|y_0^{\frac{n}{2}-1}) \cdot \mathcal{W}_{\lambda-1}^{(i)}(u_{0,o}^{2i+1}|y_{\frac{n}{2}}^{n-1}), \quad (4)$$

and $\mathcal{W}_0^{(0)}(c|y_j) = \mathbf{W}(c|y_j)$. Let us define modified log-likelihood ratios

$$S_\lambda^{(i)}(u_0^{i-1}, y_0^{n-1}) = \log \frac{\mathcal{W}_\lambda^{(i)}(u_0^{i-1}.0|y_0^{n-1})}{\mathcal{W}_\lambda^{(i)}(u_0^{i-1}.1|y_0^{n-1})}.$$

It can be seen that

$$\begin{aligned} S_\lambda^{(2i)}(u_0^{2i-1}, y_0^{N-1}) =& \max(J(0)+K(0), J(1)+K(1)) \\ &- \max(J(1)+K(0), J(0)+K(1)) \\ =& \operatorname{sgn}(a)\operatorname{sgn}(b)\min(|a|,|b|) \\ S_\lambda^{(2i+1)}(u_0^{2i}, y_0^{N-1}) =& J(u_{2i}) + K(0) - J(u_{2i}+1) - K(1) \\ =& (-1)^{u_{2i}} a + b, \end{aligned}$$

where $N = 2^\lambda$, $J(c) = \mathcal{W}_{\lambda-1}^{(i)}((u_{0,e}^{2i-1} \oplus u_{0,o}^{2i-1}).c|y_0^{\frac{N}{2}-1})$, $K(c) = \mathcal{W}_{\lambda-1}^{(i)}(u_{0,o}^{2i-1}.c|y_{\frac{N}{2}}^{N-1})$, and $a = S_{\lambda-1}^{(i)}(u_{0,e}^{2i-1} \oplus u_{0,o}^{2i-1}, y_0^{\frac{N}{2}-1})$, $b = S_{\lambda-1}^{(i)}(u_{0,o}^{2i-1}, y_{\frac{N}{2}}^{N-1})$. Then the log-likelihood of a path u_0^i can be obtained as

$$\begin{aligned} R(u_0^i|y_0^{n-1}) =& \log \mathcal{W}_m^{(i)}(u_0^i|y_0^{n-1}) \\ =& R(u_0^{i-1}|y_0^{n-1}) + \tau\left(S_m^{(i)}(u_0^{i-1}, y_0^{n-1})\right), \end{aligned} \quad (5)$$

where $R(\epsilon|y_0^{n-1}) = 0$, ϵ is an empty sequence, and

$$\tau(S,u) = \begin{cases} 0, & \operatorname{sgn}(S) = (-1)^u \\ -|S|, & \text{otherwise}. \end{cases}$$

Observe that $R(u_0^i|y_0^{n-1})$ is equal up to a sign to the approximate path metric introduced in [9]. The above derivation shows that this value is not just an approximation to the path metric used by the Tal-Vardy list decoder, but reflects the likelihood of the most probable continuation of a path in the code tree, without taking into account not-yet-processed freezing constraints.

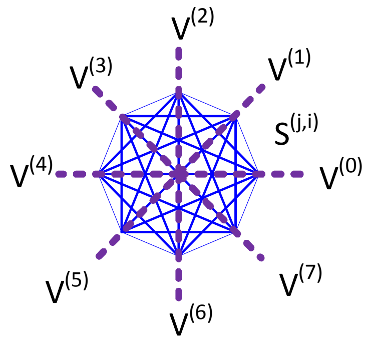

Fig. 1. The structure of freezing constraints defining a star polar subcode

III. STAR POLAR SUBCODES

We propose to generalize the construction of polar codes, so that it admits parallel decoding. A codeword of the proposed star polar subcode consists of s codewords of short polar codes, so that the information vectors of these codes have some linear dependencies. Decoding is performed by running s Tal-Vardy list decoder instances (TVLDI) for the constituent codes. The linear dependencies are exploited in order to ensure that the output vectors of TVLDIs can be combined into a valid codeword of the considered star polar subcode.

A. General case

(n,k) star polar subcode is a set of vectors

$$c = (u^{(0)}A^{(0)}, \dots, u^{(s-1)}A^{(s-1)}), \quad (6)$$

where s is a parameter called number of beams, $A^{(i)} = A_{m_i}$, $u^{(i)} \in \mathbb{F}_2^{2^{m_i}}$,

$$\begin{aligned} u^{(i)}(V^{(i)})^T =& 0, & 0 \le i < s, \quad (7) \\ u^{(j)}(S^{(j,i)})^T =& u^{(i)}(S^{(i,j)})^T, & 0 \le j < i, \quad (8) \end{aligned}$$

$n = \sum_{i=0}^{s-1} 2^{m_i}$, $V^{(i)}$ are the $\rho_i \times 2^{m_i}$ local (beam) constraint matrices, and $S^{(i,j)}$ are $\mu_{ij} \times 2^{m_i}$ cross-check matrices, such that $\mu_{ij} = \mu_{ji}$. Vector $u^{(i)}$ and subvector $u^{(i)}A^{(i)}$ of a codeword are referred to as the i-th beam and block, respectively. Figure 1 illustrates the structure of the constraints of a star polar subcode. Here dashed lines correspond to vectors $u^{(i)}$, and solid lines illustrate linear dependencies between them.

Let $\psi(P,i)$ be the position of the last non-zero entry in the i-th row of matrix P. Let $\tau_{i,p} = \psi(V^{(i)}, p), 0 \le p < \rho_i$, and let $\sigma_{i,j,t} = \psi(S^{(i,j)}, t), 0 \le t < \mu_{ij}$. It can be assumed without loss of generality that for any i and j all the values $\tau_{i,p}$ and $\sigma_{i,j,t}$ are distinct, so that the constraints (7)–(8) are linearly independent. Then the dimension of the obtained code is given by

$$k = n - \sum_{i=0}^{s-1} \rho_i - \sum_{i<j} \mu_{ij}.$$

Polar subcodes introduced in [4] are obtained by setting $s = 1$. In this case $\tau_{0,j}$ become the indices of (dynamic) frozen symbols.

Non-systematic encoding with a star polar subcode can be implemented as follows:

1) Put the payload data into positions r of $u^{(i)}$, such that $r \notin \{\tau_{i,p}|0 \leq p < \rho_i\} \cup \{\sigma_{i,j,t}|j < i, 0 \leq t < \mu_{ij}\}$.
2) For each $i \in \{0, \ldots, s-1\}$ perform the following steps in the increasing order of $\tau_{i,p}$ and $\sigma_{i,j,t}$:
   - Compute
$$u^{(i)}_{\tau_{i,p}} = \sum_{r=0}^{\tau_{i,p}-1} u^{(i)}_r V^{(i)}_{pr}, 0 \leq p < \rho_i. \tag{9}$$
   - Compute
$$u^{(i)}_{\sigma_{i,j,t}} = \sum_{r=0}^{\sigma_{i,j,t}-1} u^{(i)}_r S^{(i,j)} + \sum_{r=0}^{\sigma_{j,i,t}} u^{(j)}_r S^{(j,i)}_{tr}, \tag{10}$$
   where $0 \leq t < \mu_{ij}$.
3) Compute codeword $c = (u^{(0)}A^{(0)}, \ldots, u^{(s-1)}A^{(s-1)})$.

B. Symmetric star polar subcodes

In order to simplify code description, encoding and decoding algorithms, we suggest to impose symmetric cross-check constraints (8). That is, we propose to set $S^{(0,j)} = S^{(0,1)}, 1 \leq j < s$. Then the codewords of a symmetric star polar subcode are given by (6), where $(u^{(0)}, \ldots, u^{(s-1)})\mathbb{V}^T = 0$, and

$$\mathbb{V} = \begin{pmatrix} V^{(0)} & 0 & 0 & \ldots & 0 \\ 0 & V^{(1)} & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & V^{(s-1)} \\ \hline S^{(0,1)} & S^{(1,0)} & 0 & \ldots & 0 \\ S^{(0,1)} & 0 & S^{(2,0)} & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ S^{(0,1)} & 0 & 0 & \ldots & S^{(s-1,0)} \end{pmatrix}, \tag{11}$$

is a global constraint matrix. For brevity, matrices $S^{(0,1)}$ and $S^{(i,0)}$ will be referred to as $S^{(0)}$ and $S^{(i)}$, respectively.

Such constraint matrix can be obtained by repeated application of code construction X4 [10]. Let $H^{(j,i)}$ be check matrices of nested $(\nu_i, k_{j,i}, d_{j,i})$ linear block codes $C^{(j,i)}$, where $\nu_i = 2^{m_i}$, $j \in \{0, 1\}, 0 \leq i < s$, such that $k_{0,i} < k_{1,i}$, $k_{1,i} - k_{0,i} = k_{1,0} - k_{0,0}$ and for some matrices $\tilde{H}^{(i)}$

$$H^{(0,i)} = \begin{pmatrix} H^{(1,i)} \\ \tilde{H}^{(i)} \end{pmatrix}.$$

It can be assumed without loss of generality that the last non-zero elements of rows of each matrix

$$V^{(0,i)} = H^{(0,i)}(A^{(i)})^T = \begin{pmatrix} V^{(1,i)} \\ \tilde{V}^{(i)} \end{pmatrix}$$

are located in distinct columns p. Let $\tilde{\mathcal{F}}^{(i)}$ be the set of such integers p . We propose to obtain matrices $V^{(i)}$ and $S^{(i)}$ by appending some rows (as described in Section III-C) to matrices $V^{(1,i)}$, $\tilde{V}^{(i)}$, respectively, where $0 \leq i < s$.

Statement 1. *The minimum distance of a symmetric star polar subcode is given by*

$$d \geq \min_{0 \leq i < s} \left(d_{0,i}, \min_{j \neq i}(d_{1,i} + d_{1,j}) \right).$$

Proof: Obviously, appending any rows to matrices $V^{(1,i)}$, $\tilde{V}^{(i)}$ cannot decrease the minimum distance of the corresponding code, so these additional rows can be neglected in the proof.

Observe that the cross-check part of the global constraint matrix (11) can be transformed into a form with a $(s-1) \times (s-1)$ block-diagonal matrix with blocks $S^{(i)}$ occupying any $s-1$ block columns out of s. Let $\mathbb{A}$ be a block-diagonal matrix having $A^{(i)}$ on its diagonal. It can be seen that the check matrix $\mathbb{H} = \mathbb{V}\mathbb{A}^T$ of the obtained code has the same block structure as the constraint matrix (11).

Hence, without loss of generality one can consider a codeword having a non-zero symbol in positions $0, \ldots, 2^{m_0} - 1$, i.e. in block 0. If all symbols in other blocks are zero, then $u^{(0)}A^{(0)}$ is a codeword of $C^{(0,0)}$, so its weight is at least $d_{0,0}$. Otherwise, assume that it has a non-zero symbol in block $j > 0$. Then $u^{(0)}A^{(0)}$ and $u^{(j)}A^{(j)}$ are non-zero codewords of $C^{(1,0)}$ and $C^{(1,j)}$, so the number of non-zero symbols in the codeword is lower bounded by $d_{1,0} + d_{j,0}$. ■

C. Randomized BCH-based construction

In practice, the performance of polar codes and their generalizations depends not only on their minimum distance, but also on the probability of a list decoder killing the correct path at an early phase of the decoding process. Unfortunately, there are still no analytical tools for study and control of the latter factor. Therefore, we propose a heuristical approach to construction of codes which admit list decoding with small list size. The idea is to impose pseudo-random cross-checks (10) on symbols with sufficiently small indices $\sigma_{i,j,t}$. If the decoder diverts onto an incorrect path, then a few symbols $u^{(i)}_r$ would be different from their true values. Then (10) would be violated with high probability, and the incorrect path would get low score, and eventually killed. On the other hand, one needs to ensure that the obtained code has sufficiently high minimum distance. Hence, we propose to combine pseudo-random cross-check equations with those obtained using the above described approach based on construction X4.

Namely, we propose the following method for construction of symmetric star polar subcodes. Each matrix $V^{(i)}$ is obtained by appending to $V^{(1,i)}$ distinct weight-1 rows containing 1's in positions $t \notin \tilde{\mathcal{F}}^{(i)}$ corresponding to the least reliable (e.g. having the highest bit error probability) bit subchannels $\mathbf{W}^{(t)}_{m_i}$. Let $\hat{\mathcal{F}}^{(i)}$ be the set of such integers t, and let $f = \sum_{i=0}^{s-1} |\hat{\mathcal{F}}^{(i)}|$ be the total number of such rows appended to all $V^{(i)}$. Matrices $S^{(i)}$ are obtained by appending to $\tilde{V}^{(i)}$ χ rows containing 1's in distinct positions $p \notin \hat{F}^{(i)} \cup \tilde{F}^{(i)}$, corresponding to the least reliable bit subchannels $\mathbf{W}^{(p)}_{m_i}$, and *pseudo-random* binary values uniformly distributed over $\{0, 1\}$ in positions $h < p$. Then one has $\mu_{0,1} = k_{1,0} - k_{0,0} + \chi$, and the dimension of the obtained code satisfies

$$k = n - \sum_{i=0}^{s-1} r_{1,i} - f - (s-1)(k_{1,0} - k_{0,0} + \chi), \tag{12}$$

where $r_{1,i} = \nu_i - k_{1,i}$ is the row dimension of $H^{(1,i)}$. Pseudo-random cross-checks enable the below described decoding algorithm to quickly identify and penalize incorrect paths, reducing thus the probability of the correct path being killed at early decoding phases.

Following [4], we propose to use extended primitive narrow sense BCH codes as $C^{(j,i)}$. This ensures that all indices

$t \in \tilde{\mathcal{F}}^{(i)}$ correspond to bit subchannels $\mathbf{W}_{m_i}^{(t)}$, such that their reliability (e.g. the Bhattacharyya parameter) improves slowly while improving the reliability of the underlying channel $\mathbf{W}$. Most of such bit subchannels are unreliable.

In the case of $m_i = m, 0 \le i < s$, Statement 1 implies that one should set $d_{0,i} = d, d_{1,i} = d/2$, where d is the design minimum distance of the code. Finding optimal minimum distances $d_{r,i}, r \in \{0,1\}$ in the case of beams of different lengths remains an open problem.

Observe that one does not need to store the pseudo-random elements of $S^{(i)}$ in the encoder and decoder. One can just specify a particular algorithm and its parameters (e.g. a linear feedback shift register with some fixed seed), which can be used to recover them on demand. This enables one to specify the proposed codes in a compact way.

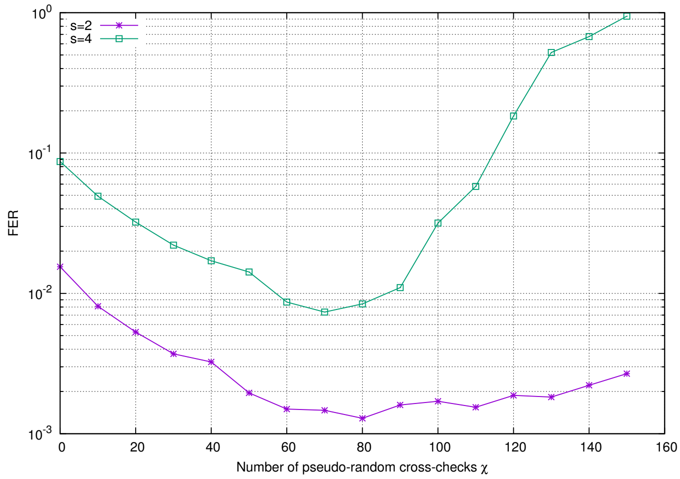

Fig. 2. Performance of $(2048, 1024, 24)$ polar subcodes with different number χ of pseudo-random cross-checks at $E_b/N_0 = 1.5$ dB

IV. Parallel list decoding of symmetric star polar subcodes

Let $Y = (y^{(0)}, \dots, y^{(s-1)})$ be the noisy vector obtained by transmitting a codeword (6) of a symmetric star polar subcode over a memoryless channel. The objective of the decoder is to find $(u^{(0)}, \dots, u^{(s-1)})$ with the highest possible log-likelihood

$$R(u^{(0)}, \dots, u^{(s-1)}|Y) = \sum_{i=0}^{s-1} \log \mathbf{W}_{m_i}^{(\nu_i - 1)}(u^{(i)}|y^{(i)}),$$

subject to constraints (7)–(8).

In this section we present a parallel version of the Tal-Vardy list decoding algorithm for symmetric star polar subcodes. For the sake of simplicity we consider the case of $\nu_i = \nu, 0 \le i < s$, but the proposed approach can be extended to the case of beams of different size. The idea of the below described algorithm is to construct at each phase ϕ lists of possible partial vectors $u^{(i)}$ (i.e. their first ϕ elements), and compute for each such partial vector an upper bound on the final value of $\mathbf{W}_{m_i}^{(\nu_i-1)}(u^{(i)}|y^{(i)})$. The cross-check equations (10) are used at each phase in order to keep only those partial vectors, which may correspond to a valid codeword of the considered code.

The proposed decoding algorithm makes use of s Tal-Vardy list decoder instances (TVLDI), which perform LLR calculation and memory management. The i-th instance takes as input $y^{(i)}$ and constructs at most L candidates for $u^{(i)}$ satisfying (7). Let $U_{\phi,j}^{(i)} \in \mathbb{F}_2^{\phi+1}, 0 \le j < L$, be the paths considered by the i-th TVLDI at phase ϕ.

At phase ϕ each TVLDI computes *local score* $R(U_{\phi,j}^{(i)}|y^{(i)})$ given by (5) for each of the considered paths $U_{\phi,j}^{(i)}$ as described in Section II-B. Recall, that $R(U_{\phi,j}^{(i)}|y^{(i)})$ is an upper bound on $\log \mathbf{W}_{m_i}^{(\nu_i-1)}(u^{(i)}|y^{(i)})$ over all $u^{(i)}$, such that their first $\phi+1$ elements are equal to those of $U_{\phi,j}^{(i)}$.

If $\phi \in \{\tau_{i,j}\}$ for some i, then each path $U_{\phi,j}^{(i)}$ is extended with its frozen value, which is computed according to (9). Otherwise, the paths $U_{\phi,j}^{(i)}$ are cloned and TVLDI synchronization is performed. Synchronization consists in selecting those paths, which may be combined into a valid codeword of a symmetric star polar subcode satisfying all constraints (9)–(10).

Let $g(i,\phi) = \{t|\sigma_{i,0,t} \le \phi \wedge \sigma_{0,i,t} \le \phi\}$ be the set of cross-check equations (10), which can be evaluated at phase ϕ. Let

$$z_{i,\phi}(U_{\phi,j}^{(i)}) = U_{\phi,j}^{(i)}(S^{(i)})^T_{g(i,\phi),\phi}$$

be the *partial cross-check vector* of path $U_{\phi,j}^{(i)}$, where $(Q)_{B,\phi}$ is a submatrix of Q consisting of rows given by set B and columns $0, \dots, \phi$, . Let us further define *matching score*

$$\beta_{i,\phi}(x|y^{(i)}) = \max_{\substack{0 \le j < L \\ z_{i,\phi}(U_{\phi,j}^{(i)}) = x}} R(U_{\phi,j}^{(i)}|y^{(i)}), x \in \mathbb{F}_2^{|g(i,\phi)|}.$$

Here maximization is performed over the set of paths constructed by the i-th TVLDI at phase ϕ, having a particular value of the partial cross-check vector. It can be seen that $\beta_{i,j}(x|y^{(i)})$ is an upper bound on the log-likelihood of paths in the code tree with partial cross-check vector x. If there is no path in the i-th TVLDI with $z_{i,\phi}(U_{\phi,j}^{(i)}) = x$, then we assume $\beta_{i,\phi}(x|y^{(i)}) = -\infty$.

If $|g(i,\phi)|$ is a small value, then all values $\beta_{i,\phi}(x|y^{(i)})$ can be computed from $R(U_{\phi,j}^{(i)}|y^{(i)})$ with complexity $O(L)$ via counting sort techniques. Otherwise, a generic sorting algorithm with complexity $O(L \log L)$ or hashing techniques can be used.

The *global score* of $U_{\phi,j}^{(i)}$ is defined as an upper bound on the log-likelihood of a codeword, which can be obtained by combining $U_{\phi,j}^{(i)}$ with the paths $U_{\phi,j'}^{(i')}$ obtained by all other TVLDIs $i' \neq i$, i.e.

$$G_i^{(\phi)}(U_{\phi,j}^{(i)}|Y) = R(U_{\phi,j}^{(i)}|y^{(i)}) + \sum_{i' \neq i} \beta_{i',\phi}\left(z_{i,\phi}\left(U_{\phi,j}^{(i)}\right)|y^{(i')}\right). \quad (13)$$

We propose to use $G_i^{(\phi+1)}(u_0^\phi|Y)$ instead of $R(U_{\phi,j}^{(i)}|y^{(i)})$ in each of TVLDI for path sorting and killing. The remaining implementation details are the same as in [3].

After all TVLDI's have processed the final phase $\nu - 1$, a codeword (6) of the symmetric star polar subcode with the highest possible $R(u^{(0)}, \dots, u^{(s-1)}|Y)$ should be constructed from obtained lists of $U_{\phi,j}^{(i)}$, taking into account constraints (8). This task can be solved by finding the best cross-check value

$$\hat{x} = \arg\max_x \sum_{i=0}^{s-1} \beta_{i,\nu-1}(x|y^{(i)}),$$

so that one can select

$$u^{(i)} = \arg \max_{j: z_{i,\nu-1}(U_{\phi,j}^{(i)}) = \hat{x}} R(U_{\phi,j}^{(i)}|y^{(i)}).$$

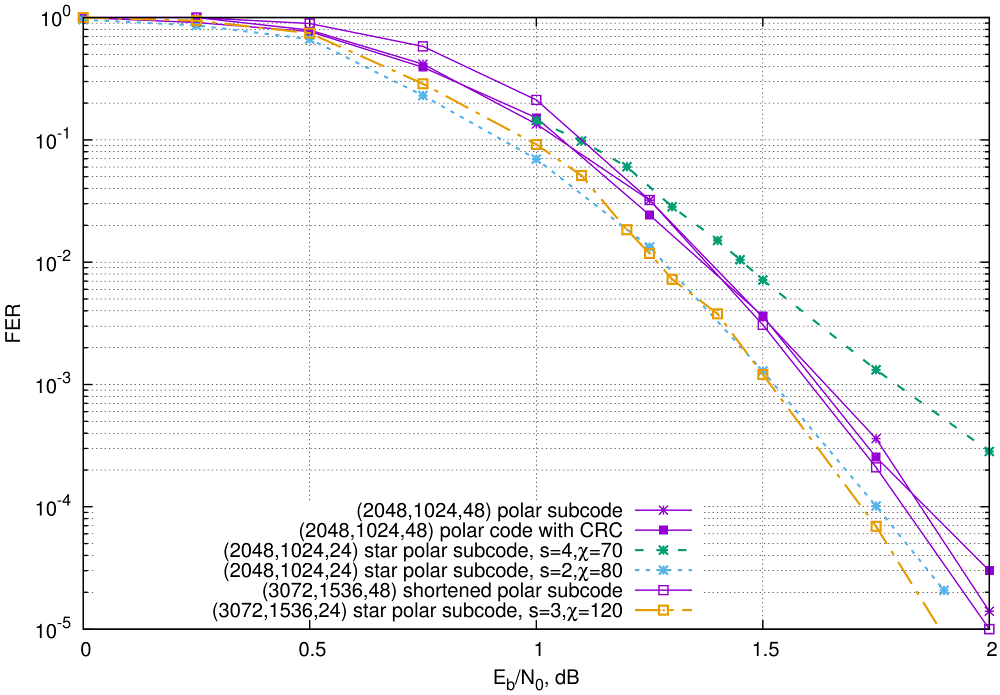

Fig. 3. Performance of star polar subcodes

The proposed decoding algorithm requires $O(sL\nu\log\nu)$ operations for computing local path scores $R(U_{\phi,j}^{(i)}|y^{(i)})$, and $O(s\nu L\log L)$ operations for computing global scores and sorting the paths, i.e. its overall complexity still remains $O(Ln\log n)$, similarly to the Tal-Vardy list decoding algorithm. However, the main advantage of the proposed algorithm is that it allows the decoding to be performed in parallel using s synchronized TVLDIs, where synchronization is performed by exchanging the matching scores. Each of the TVDIs performs $\nu = n/s$ steps, while list decoding of a conventional polar (sub)code requires n steps. This results in significant reduction of the decoding latency. The overhead of synchronization can be reduced by employing block processing techniques considered in [11], [12].

V. Numeric results

The performance of symmetric star polar subcodes was studied in the case of AWGN channel and BPSK modulation. Size of the list in the proposed decodin algorithm was set to $L = 32$.

Figure 2 illustrates the performance of star polar subcodes with different number of pseudo-random cross-checks χ at $E_b/N_0 = 1.5$ dB. It can be seen that for each s there exists an optimal value χ_s of the number of pseudo-random cross-checks. If $\chi < s$, then the TVLDIs with high probability kill the correct paths before reaching the phase where cross-checks obtained from the check matrices of the extended BCH codes. If $\chi > \chi_s$, then obtaining a code with target dimension k requires one to reduce f in (12), i.e. keep unfrozen many symbols corresponding to unreliable bit subchannels, also increasing the probability of correct paths being killed by some TVLDI at early steps of the decoding algorithm.

Figure 3 provides performance comparison of star polar subcodes, polar subcodes of extended BCH codes [4], i.e. star polar subcodes with $s = 1$ beam, and a polar code with CRC. It can be seen that star polar subcodes with $s = 2, 3$ outperform those with $s = 1$. However, for $s = 4$ there is a noticeable performance loss. This is the price one should pay for 4-times reduction of the decoding latency. It may be reduced by developing more efficient TVLDI synchronization techniques.

Observe also that we had to reduce the design minimum distance in the case of $s > 1$ compared to $(2048, 1024, 48)$ polar subcode. According to (12), this is needed in order to keep the total number of cross-checks low, so that the symbols corresponding to unreliable bit subchannels are frozen.

In all simulations in almost all cases of decoding error the obtained codewords were less likely than the transmitted one, i.e. the errors were caused by the suboptimality of the decoding algorithm.

VI. Conclusions

In this paper star polar subcodes were introduced. The proposed construction together with the proposed generalization of the Tal-Vardy list decoding algorithm allows one to reduce the decoding latency by a factor of s, where s is the number of beams in the star code, while keeping the overall decoding complexity $O(Ln\log n)$, similarly to the case of list decoding of conventional polar codes. It was shown that in the case of $s = 2, 3$ the symmetric star polar subcodes provide slight performance gain with respect to non-star polar subcodes. However, for $s \geq 4$ the performance of the obtained codes is not as good. Further research on code construction techniques in this case is needed.

An essential ingredient of the proposed code construction is a set of pseudo-random cross-checks, which need to be provided in order to prevent the decoder from killing the correct path at early phases of the decoding process. Any progress in theoretical understanding of list decoding of polar codes may lead to better techniques for construction of such cross-checks.

The proposed construction was inspired by the concept of star trellis [13], [14]. It may be an interesting research problem to employ the construction of known codes having star trellises in order to obtain efficiently decodable star polar subcodes.

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