Design of Polar Codes for Rayleigh Fading Channel

Peter Trifonov

Peter the Great Saint-Petersburg State Polytechnic University

Email: petert@dcn.icc.spbstu.ru

Abstract—**A simple method for construction of polar codes for Rayleigh fading channel is presented. The subchannels induced by the polarizing transformation are modeled as multipath fading channels, and their diversity order and noise variance are tracked. It is shown that the performance of polar codes in fading channels can be substantially improved by employing the construction with dynamic frozen symbols. The obtained codes are shown to provide significant gain with respect to LDPC codes.**

I. Introduction

Polar codes were recently shown to be able to achieve the capacity of binary input memoryless output-symmetric channels (BMSC), while having low complexity encoding and decoding algorithms [1]. Furthermore, there exist polynomial complexity algorithms for their construction.

In order to construct a polar code for a specific channel, one needs to investigate the performance of bit subchannels induced by the polarizing transformation. Typically, one is interested in bit error probability for these subchannels. This information is needed not only for a code designer, but also for sequential decoding algorithms [2], [3], which were shown to have very low average complexity and provide near-maximum-likelihood performance.

For any BMSC, subchannel error probabilities can be computed with complexity $O(n\mu^2 \log \mu)$ using an implementation of density evolution suggested in [4]. However, this algorithm is quite difficult to implement, and requires high number μ of quantization levels in order to achieve reasonable precision. In the case of AWGN channel, Gaussian approximation provides very good accuracy and has complexity $O(n)$ [5], [6].

For other channels the problem of simple and accurate estimation of subchannel error probability is still largely unsolved. Possible approaches include tracking lower and upper bounds on Bhattacharyya parameters of the bit subchannels [7]. However, these bounds are not guaranteed to be tight. Fading channels with two states were considered in [8], and some code constructions for this case were proposed. It was suggested in [9] to approximate a fading AWGN channel by a fading binary symmetric channel. Monte-Carlo simulations were suggested in [10] in order to find good subchannels in the case of the Rayleigh fading channel with known channel distribution. However, neither of these works provide simple techniques to evaluate the bit error probability of the subchannels induced by the polarizing transformation.

The practical performance of moderate length polar codes appears to be much worse compared to the existing turbo and LDPC codes. This is both due to suboptimality of the successive cancellation decoding algorithm, and poor minimum distance of polar codes. The first problem can be mitigated by employing list decoding techniques [11] and their sequential generalizations [12], [2], [3]. Solving the second problem requires one to introduce some dependencies between information symbols of polar codes. This can be done either by pre-coding the data with an error-correcting or error-detecting code [6], [4], or by ensuring that the polar code is a subcode of an algebraic code with sufficiently high minimum distance [2].

In this paper we present a simple method for construction of moderate length polar codes for independent Rayleigh fading channel. The proposed approach models the bit subchannels induced by the polarizing transformation as fading channels. The parameters of these channels satisfy recurrent equations, which can be easily solved numerically.

The paper is organized as follows. In Section II some background on polar codes is presented. The proposed method for finding the error probability in bit subchannels of the polarizing transformation in the case of Rayleigh fading channel, and construction of the corresponding polar codes is described in Section III. Numeric results illustrating the accuracy of the proposed approximation and performance of polar codes under successive cancellation and sequential decoding are provided in Section IV. Finally, some conclusions are drawn.

II. Background

A. Channel polarization

Consider a binary input memoryless output symmetric channel with transition probabilities $W(y|x), x \in \mathcal{X}, y \in \mathcal{Y}$, where $\mathcal{X}, \mathcal{Y}$ are input and output alphabets, respectively. The synthetic bit subchannels $W_m^{(i)}, 0 \le i < 2^m$ of the polarizing transformation are given by

$$W_{m+1}^{(2i)}(y_0^{2n-1}, u_0^{2i-1}|u_{2i})$$
$$= \frac{1}{2} \sum_{u_{2i+1}=0}^{1} W_m^{(i)}(y_0^{n-1}, u_{0,e}^{2i-1} + u_{0,o}^{2i-1}|u_{2i} + u_{2i+1}) \quad (1)$$
$$\cdot W_m^{(i)}(y_n^{2n-1}, u_{0,o}^{2i-1}|u_{2i+1})$$

$$W_{m+1}^{(2i+1)}(y_0^{2n-1}, u_0^{2i}|u_{2i+1})$$
$$= \frac{1}{2} W_m^{(i)}(y_0^{n-1}, u_{0,e}^{2i-1} + u_{0,o}^{2i-1}|u_{2i} + u_{2i+1}) \quad (2)$$
$$\cdot W_m^{(i)}(y_n^{2n-1}, u_{0,o}^{2i-1}|u_{2i+1}),$$

where $n = 2^m$, $W_0^{(0)}(y|x) = W(y|x)$, and $a_i^j = (a_i, \ldots, a_j)$. Here m and i denote layer and phase, respectively. It is possi-

ble to show that the capacities of these subchannels converge with m to 0 or 1, and the fraction of subchannels with capacity close to 1 converges to the capacity of the original channel $W(y|x)$ [1]. Let $\mathcal{F}$ be the set of low-capacity subchannels. Then one can perform almost error-free communication by encoding the data as

$$x_0^{n-1} = u_0^{n-1} \underbrace{B_m \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{\otimes m}}_{A_m},$$

and transmitting $x_i, 0 \leq i < 2^m - 1$, over the channel $W(y|x)$, where B_m is the $2^m \times 2^m$ bit-reversal permutation matrix, $u_i = 0, i \in \mathcal{F}$ (frozen symbols), and the remaining values u_i are set to the payload data.

At the receiver side, one can successively estimate $\widehat{u}_i = \arg\max_{u_i \in \mathbb{F}_2} W_m^{(i)}(y_0^{n-1}, u_0^{i-1} | u_i), i \notin \mathcal{F}$, assuming that the correct values of $u_j, j < i$, are available. This is known as the successive cancellation decoding algorithm. Alternatively, decoding can be implemented using the log-likelihood ratios $L_m^{(i)}(y_0^{n-1}, u_0^{i-1}) = \ln \frac{W_m^{(i)}(y_0^{n-1}, u_0^{i-1} | u_i = 0)}{W_m^{(i)}(y_0^{n-1}, u_0^{i-1} | u_i = 1)}$, which can be recursively computed as

$$L_m^{(2i)}(y_0^{2n-1}, u_0^{2i-1})$$
$$= 2 \tanh^{-1}(\quad \tanh(L_m^{(i)}(y_0^{n-1}, u_{0,e}^{2i-1} + u_{0,o}^{2i-1})/2)$$
$$\cdot \tanh(L_m^{(i)}(y_n^{2n-1}, u_{0,e}^{2i-1})/2)), \quad (3)$$
$$L_{m+1}^{(2i+1)}(y_0^{2n-1}, u_0^{2i}) = L_m^{(i)}(y_n^{2n-1}, u_{0,o}^{2i-1})$$
$$+ (-1)^{u_{2i}} L_m^{(i)}(y_0^{n-1}, u_{0,e}^{2i-1} + u_{0,o}^{2i-1}), \quad (4)$$

The standard way to construct a moderate length (n, k) polar code is to compute error probabilities of bit subchannels $W_m^{(i)}(y_0^{n-1}, u_0^{i-1} | u_i)$, and construct $\mathcal{F}$ as the set of $n - k$ subchannels with the highest error probability. Assuming that zero codeword was transmitted, this can be done by recursively computing the distributions of $L_m^{(i)}(y_0^{n-1}, \mathbf{0})$ values, which are related to $L_0^{(0)}(y)$ via (3)–(4), using density evolution [13], [4]. For the case of AWGN channel, these distributions are quite close to the Gaussian one, and they can be characterized by a single parameter (mean value), which can be easily computed. This greatly simplifies the calculations [6].

B. Polar codes with dynamic frozen symbols

The successive cancellation algorithm fails to provide maximum likelihood decoding. However, its list and sequential extensions were shown to provide near-optimal performance for sufficiently large list size [14], [12], [2], [3]. This performance yet appears to be quite poor due to very low minimum distance of polar codes. To address this problem, it was suggested in [2] to set some frozen input symbols u_i of the polarizing transformation not to 0, as in the above described construction, but to some linear combinations of other symbols u_i. These linear combinations should be constructed in such way, so that the obtained polar code has sufficiently high minimum distance.

Namely, let H be a check matrix of a $(2^m, K, d)$ code, called parent code. Let $V = QHA_m^T$, where $(2^m - K) \times (2^m - $

$K)$ invertible matrix Q is selected in such way, so that the rows of matrix V end[1] in distinct positions $j_i, 0 \leq i < 2^m - K$. Hence, one obtains

$$u_{j_i} = \sum_{s=0}^{j_i - 1} V_{i,s} u_s.$$

Such symbols u_{j_i} are called dynamic frozen symbols. Let $\mathcal{F}' = \{j_i | 0 \leq i < 2^m - K\}$. In order to obtain a $(2^m, k, d), k \leq K$, polar code with dynamic frozen symbols, one should impose additional constraints $u_i = 0, i \in \mathcal{F}''$, where $\mathcal{F}''$ is the set of $K - k$ indices of bit subchannels with the highest error probabilities, which are not included into $\mathcal{F}'$. It was suggested in [2] to use extended BCH codes as parent codes in this construction.

Observe that the set $\mathcal{F} = \mathcal{F}' \cup \mathcal{F}''$ obtained in this way is not guaranteed to consist of subchannels with the highest error probability. Therefore, the performance of polar codes with dynamic frozen symbols under successive cancellation decoding may be worse than that of the corresponding classical polar codes. However, this construction was shown to provide significant gain with respect to classical polar codes, polar codes with CRC, and LDPC codes of moderate length, provided that list or sequential decoding is used.

III. χ APPROXIMATION FOR SUBCHANNELS OF THE POLARIZING TRANSFORMATION

A. System model

Let us consider the Rayleigh fading channel, where the received symbols are given by

$$y_j = h_j (-1)^{x_j} + \eta_j, \quad (5)$$

where h_j follow the Rayleigh distribution, and $\eta_i \sim \mathcal{N}(0, \sigma^2)$. Here values h_j and η_j are assumed to be independent. Rayleigh distribution is a special case of χ distribution with $\nu = 2$ degrees of freedom, which is given by probability density function

$$\psi(x, \nu) = \frac{2x^{\nu-1} e^{-x^2}}{\Gamma(\nu/2)}. \quad (6)$$

Observe that here ν does not need to be an integer.

The log-likelihood ratios of the received symbols are given by $L_0^{(0)}(y_j) = z_j = \frac{2h_j y_j}{\sigma^2}$. Assuming zero codeword transmission, one obtains that the conditional probability density function of received symbol LLRs is given by [15]

$$f_{LLR}(z | x_j = 0) = \frac{\sigma^2}{2\sqrt{1 + 2\sigma^2}} e^{\frac{z - |z|\sqrt{1+2\sigma^2}}{2}} \quad (7)$$

B. Two-dimensional method

The distribution given by (7) is quite different from the Gaussian one. Therefore, Gaussian approximation fails to correctly predict the bit error rate in the subchannels induced by the polarizing transformation. In this section we present an alternative way to model these subchannels.

---

[1]Row i ends in column j_i if $V_{i,j_i} = 1, V_{i,s} = 0, s > j_i$.

m \ i	0	1	2	3	4	5	6	7
0	$(1, 0.5)$							
1	$(1.03, 0.8)$	$(2, 0.5)$						
2	$(1.07, 1.4)$	$(2.06, 0.8)$	$(1.95, 0.63)$	$(4, 0.5)$				
3	$(1.11, 3.05)$	$(2.14, 1.4)$	$(1.77, 1.01)$	$(4.12, 0.8)$	$(1.81, 0.8)$	$(3.9, 0.63)$	$(4.04, 0.57)$	$(8, 0.5)$

Observe that (4) can be considered as an implementation of the maximum ratio combining, a technique used to combine multiple received instances of the same signal [16]. Given received noisy instances $r_j = h_j s + \eta_j, 0 \le j < \Lambda$, of the same transmitted symbol $s \in \{-1, 1\}$, one can estimate it as

$$\widehat{s} = \sum_{j=0}^{\Lambda-1} h_j r_j = \underbrace{\left(\sum_{j=0}^{\Lambda-1} h_j^2 \right)}_{\tilde{h}^2} s + \underbrace{\sum_{j=0}^{\Lambda-1} h_j \eta_j}_{\eta} .$$

If h_j follow χ distribution with 2 degrees of freedom, then $\tilde{h}^2$ follows χ^2 distribution with 2Λ degrees of freedom, and η follows normal distribution with variance $\tilde{h}^2 \sigma^2$. It is possible to show that the probability of error provided by this method is given by [16]

$$P(\Lambda, \sigma^2) = \left(\frac{1-\mu}{2} \right)^\Lambda \sum_{k=0}^{\Lambda-1} \binom{\Lambda-1+k}{k} \left(\frac{1+\mu}{2} \right)^k$$
$$= 1 - \left(\frac{1-\mu^2}{4} \right)^\Lambda {}_2F_1 \begin{bmatrix} 1 & 2\Lambda \\ \Lambda+1 \end{bmatrix}; \frac{1+\mu}{2} \right] \binom{2\Lambda-1}{\Lambda} \tag{8}$$

where $\mu = \frac{1}{\sqrt{1+2\sigma^2}}$, and ${}_2F_1 \begin{bmatrix} a & b \\ c \end{bmatrix}; z \right]$ is the generalized hypergeometric function. Efficient techniques for its evaluation are described in [17], and their implementation is available in various numeric libraries. For sufficiently small σ, this expression can be approximated by

$$P(\Lambda, \sigma^2) \approx \left(\frac{\sigma^2}{2} \right)^\Lambda \binom{2\Lambda-1}{\Lambda}. $$

Given the availability of a relatively simple expression for bit error probability of a maximum ratio combining receiver, we propose to model the subchannels of the polarizing transformation as in (5), where h_j follows χ distribution with $2\Lambda_m^{(i)}$ degrees of freedom, and $\eta_i \sim \mathcal{N}(0, (\sigma_m^{(i)})^2)$, where m and i are layer and phase indices, respectively, and $\Lambda_0^{(0)} = 1, \sigma_0^{(0)} = \sigma$. That is, each subchannel is characterized by a pair $(\Lambda_m^{(i)}, \sigma_m^{(i)})$. It can be seen from (4) that

$$\Lambda_{m+1}^{(2i+1)} = 2\Lambda_m^{(i)} \tag{9}$$
$$\sigma_{m+1}^{(2i+1)} = \sigma_m^{(i)}. \tag{10}$$

It appears that the channel obtained by combining two channels with parameters (Λ, σ) according to (1) or (3) can be also well approximated by a similar channel with some other
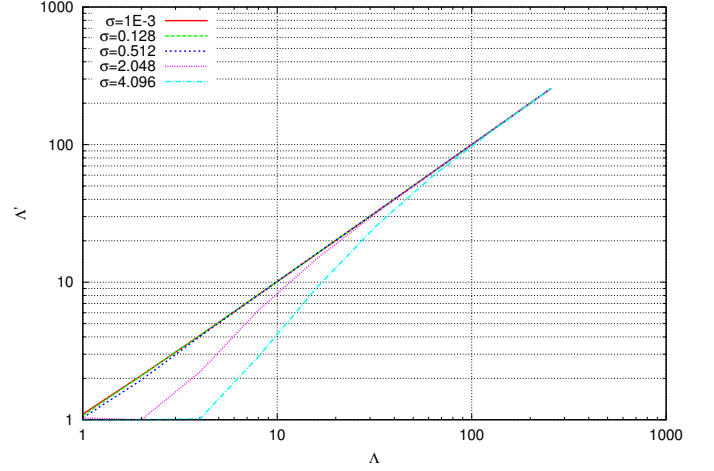

Fig. 1. $\Lambda' : C_t(\Lambda', \sigma') = C_t(\Lambda, \sigma)^2$

parameters (Λ', σ'). Assume that zero codeword is transmitted. For some layer m and phase i, let

$$C_t(\Lambda_m^{(i)}, \sigma_m^{(i)}) = E\left[\tanh^t \left(\frac{L_m^{(i)}}{2} \right) \right] = E\left[\tanh^t \left(\frac{hy}{(\sigma_m^{(i)})^2} \right) \right]$$
$$= \int_0^\infty \int_{-\infty}^\infty \tanh^t \left(\frac{hy}{(\sigma_m^{(i)})^2} \right) \psi(h, 2\Lambda_m^{(i)}) \phi(y-h, \sigma_m^{(i)}) dy dh,$$

where $\phi(y, \sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{y^2}{2\sigma^2}}$, $t \in \{1, 2\}$. Then from (3) one obtains

$$C_t(\Lambda_{m+1}^{(2i)}, \sigma_{m+1}^{(2i)}) = C_t(\Lambda_m^{(i)}, \sigma_m^{(i)})^2, t = 1, 2. \tag{11}$$

This system of equations can be solved numerically to obtain the parameters of a subchannel at layer $m+1$ and phase $2i$. Observe that this may result in non-integer values of $\Lambda_{m+1}^{(2i)}$.

Example 1. Table I illustrates the parameters of the bit subchannels for the case of 8×8 polarizing transformation and $\sigma = 0.5$.

C. A simplified one-dimensional method

Solving (11) is a computationally intensive problem, since this involves multiple two-dimensional numeric integration. Therefore we propose a simplified approach.

Figure 1 illustrates the dependency of Λ' on Λ, such that $C_t(\Lambda', \sigma') = C_t(\Lambda, \sigma)^2, t = 1, 2$. It can be seen that for sufficiently small σ one has $\Lambda' \approx \Lambda$. Hence, we propose to set $\Lambda_{m+1}^{(2i)} = \Lambda_m^{(i)}$, and compute just $\sigma_{m+1}^{(2i)}$. Let $p_{m,i} =$

$m \backslash i$	0	1	2	3	4	5	6	7
0	0.5							
1	0.79	0.5						
2	1.43	0.79	0.64	0.5				
3	3.51	1.42	1.12	0.79	0.86	0.64	0.57	0.5
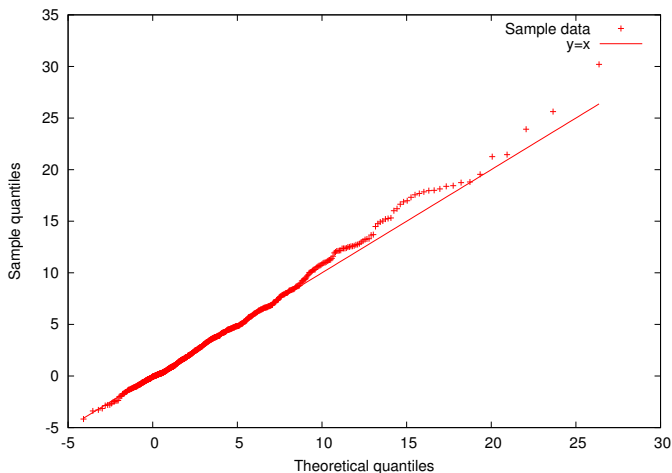
Fig. 2. Quantile-quantile plot for $L_1^{(0)}$

Fig. 3. Error probabilities in bit subchannels for $m = 7, \sigma = 0.5$

$P(\Lambda_m^{(i)}, \sigma_m^{(i)})$ be the error probability in subchannel $W_m^{(i)}(y|x)$. It is possible to show [18], [19] that $p_{m+1,2i} = 2p_{m,i} - 2p_{m,i}^2$. Having obtained $p_{m+1,2i}$, one can find appropriate $\sigma_{m+1}^{(2i)}$ by solving numerically just a single equation (8). That is, one obtains the following recursion for $\sigma_m^{(i)}$:

$$P(\Lambda_{m+1}^{(2i)}, \sigma_{m+1}^{(2i)}) = 2P(\Lambda_m^{(i)}, \sigma_m^{(2i)})(1 - P(\Lambda_m^{(i)}, \sigma_m^{(i)})). \quad (12)$$

Observe that in this case one has

$$\Lambda_m^{(i)} = 2^{\mathrm{wt}(i)}, \quad (13)$$

where $\mathrm{wt}(i)$ is the number of non-zero bits in the binary expansion of integer i. The parameters of the subchannels with odd indices are still given by (9)–(10).

Example 2. Table II presents the approximate values of $\sigma_m^{(i)}$ for the case of 8-dimensional polarizing transformation and $\sigma = 0.5$. It can be seen that these values are quite close to those given in Table I.

The presented results show that $\Lambda_1^{(0)}$, which is obtained from the received noisy values by a single application of (3), can be considered as a random variable with the probability density function given by (7) with $\sigma = 0.8$. Figure 2 presents the corresponding QQ plot. It can be seen that the sample data indeed follows this distribution quite closely.

D. Code construction

The above described techniques can be used to estimate the error probabilities in bit subchannels of the polarizing transformation. To obtain a $(2^m, k)$ polar code, one should construct
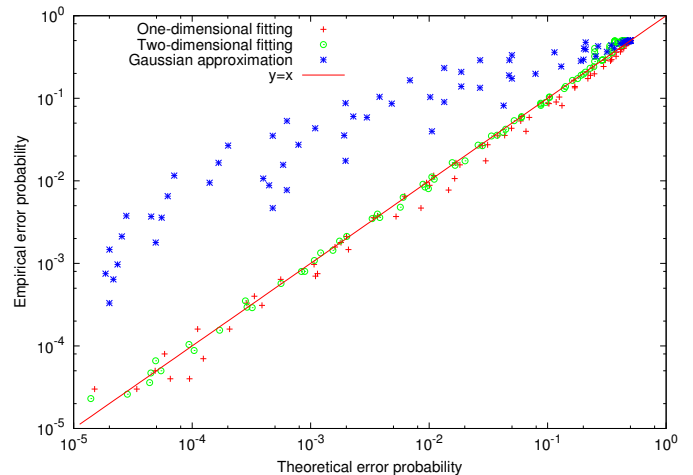
$\mathcal{F}$ as the set of indices of the subchannels with the highest error probability. However, the decoding error probability of error correcting codes in fading channels depends strongly on their minimum distance. Hence, in order to obtain reasonable performance, one should employ the construction described in Section II-B, and use a sequential or a list decoding algorithm.

IV. NUMERIC RESULTS

Figure 3 illustrates theoretical vs. empirical error probabilities in bit subchannels of 128×128 polarizing transformation A for the case of $\sigma = 0.5$. Both the two-dimensional method for fitting distribution parameters, which is given by (9)–(11), and the one-dimensional method given by (10), (12), (13), are considered. Additionally, the results obtained with Gaussian approximation [19] are provided. It can be seen that the two-dimensional method provides quite accurate results in all except very bad subchannels, where finding numerically a solution of (11) poses a major challenge. The one-dimensional method slightly overestimates the error probability in good subchannels. Observe that this does not have an adverse effect on code construction. It can be also seen that the Gaussian approximation does not work well for polar codes over the Rayleigh fading channel.

Figure 4 illustrates the performance of $(1024, 512, d)$ polar codes under the successive cancellation (SC) and sequential decoding [3] with different values of list size l. The results are reported for the case of codes optimized for Rayleigh fading channel with $E_b/N_0 = 5dB$, and AWGN channel with $E_b/N_0 = 2dB$. These codes are designated with subscripts R and F, respectively. One-dimensional method was used to evaluate the error probabilities of bit subchannels in the case of Rayleigh fading channel while designing the codes. The code with minimum distance $d = 16$ is the classical Arikan polar code, and codes with minimum distance $d = 24$ and $d = 28$ are polar codes with dynamic frozen symbols. For comparison, the performance of a $(1032, 516)$ LDPC code is shown. In the latter case, belief propagation decoding with up to 200
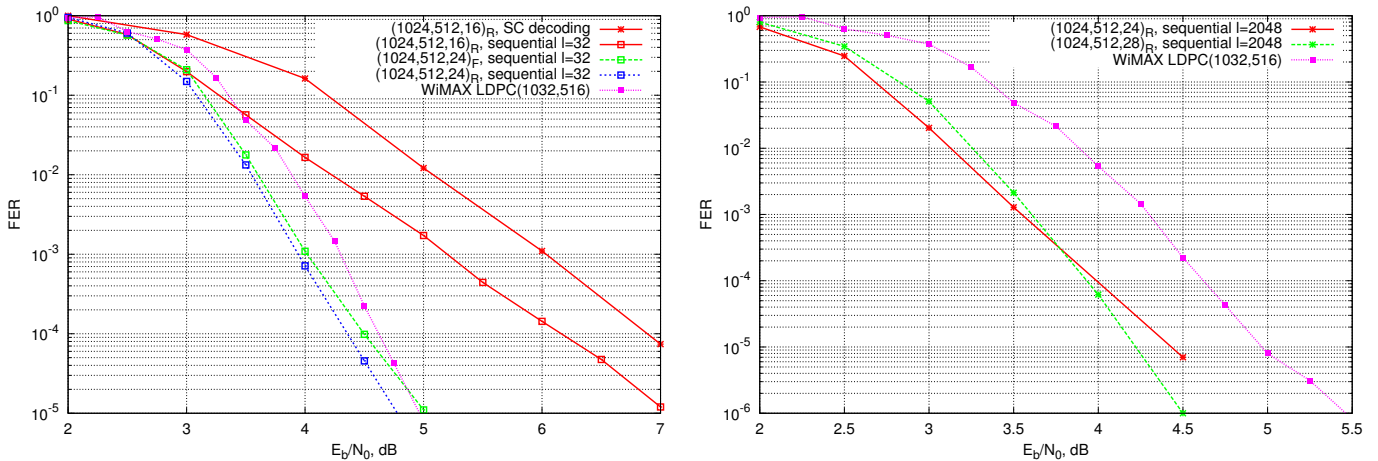
Fig. 4. Performance of polar codes

iterations was used. It can be seen that the performance of classical polar codes is quite bad, even if sequential decoding is used. Polar codes with dynamic frozen symbols provide much better performance. Observe that the code optimized for the Rayleigh fading channel does outperform the one optimized for AWGN channel. It can be seen that for sufficiently large l the considered polar codes provide 0.8 dB gain with respect to the LDPC code. At $E_b/N_0 = 4dB$ the average decoding complexity for the $(1024, 512, 28)_R$ code with $l = 2048$ was $2.8 \cdot 10^4$ summations and $4.9 \cdot 10^4$ comparisons, while for the LDPC code at $E_b/N_0 = 4.8$ dB the average number of summations and evaluations of $\log \tanh(x/2)$ was $1.4 \cdot 10^5$ and $4.8 \cdot 10^4$, respectively. This justifies fairness of the comparison.

V. Conclusions

In this paper a method for construction of polar codes for Rayleigh fading channel was presented. The subchannels induced by the polarizing transformation are modeled as fading ones, where fading gain follows χ distribution. This enables one to characterize these subchannels by just two parameters, which can be computed using numerical techniques. An additional approximation was introduced, which reduces the number of parameters to be tracked to one, but results in slightly overestimated error probability estimates. A simple and accurate method for computing the error probabilities in bit subchannels is needed not only for designing codes, but also for low-complexity sequential decoding algorithm [3].

It was shown that classical polar codes provide quite bad performance in the fading channel. However, by employing the construction with dynamic frozen symbols the performance can be improved dramatically, so that the obtained codes provide 0.8 dB gain with respect to a similar LDPC code.

References

[1] E. Arikan, "Channel polarization: A method for constructing capacity-achieving codes for symmetric binary-input memoryless channels," *IEEE Transactions on Information Theory*, vol. 55, no. 7, pp. 3051–3073, July 2009.

[2] P. Trifonov and V. Miloslavskaya, "Polar codes with dynamic frozen symbols and their decoding by directed search," in *Proceedings of IEEE Information Theory Workshop*, September 2013, pp. 1 – 5.

[3] V. Miloslavskaya and P. Trifonov, "Sequential decoding of polar codes," *IEEE Communications Letters*, vol. 18, no. 7, pp. 1127–1130, 2014.

[4] I. Tal and A. Vardy, "How to construct polar codes," *IEEE Transactions on Information Theory*, vol. 59, no. 10, pp. 6562–6582, October 2013.

[5] S.-Y. Chung, T. J. Richardson, and R. L. Urbanke, "Analysis of sum-product decoding of low-density parity-check codes using a Gaussian approximation," *IEEE Transactions on Information Theory*, vol. 47, no. 2, February 2001.

[6] P. Trifonov, "Efficient design and decoding of polar codes," *IEEE Transactions on Communications*, vol. 60, no. 11, pp. 3221 – 3227, November 2012.

[7] Y. Zhang, A. Liu, K. Pan, C. Gong, and S. Yang, "A practical construction method for polar codes," *IEEE Communications Letters*, vol. 18, no. 11, November 2014.

[8] J. Boutros and E. Biglieri, "Polarization of quasi-static fading channels," in *Proceedings of IEEE International Symposium on Information Theory*, 2013.

[9] H. Si, O. O. Koyluoglu, and S. Vishwanath, "Polar coding for fading channels: Binary and exponential channel cases," *IEEE Transactions On Communications*, vol. 62, no. 8, August 2014.

[10] A. Bravo-Santos, "Polar codes for the Rayleigh fading channel," *IEEE Communications Letters*, vol. 17, no. 12, December 2013.

[11] I. Tal and R. Roth, "On list decoding of alternant codes in the Hamming and Lee metrics," in *Proceedings of IEEE International Symposium on Information Theory*, 2003.

[12] K. Chen, K. Niu, and J. Lin, "Improved successive cancellation decoding of polar codes," *IEEE Transactions on Communications*, vol. 61, no. 8, pp. 3100–3107, August 2013.

[13] R. Mori and T. Tanaka, "Performance of polar codes with the construction using density evolution," *IEEE Communications Letters*, vol. 13, no. 7, July 2009.

[14] I. Tal and A. Vardy, "List decoding of polar codes," in *Proceedings of IEEE International Symposium on Information Theory*, 2011, pp. 1–5.

[15] S. ten Brink, "Convergence behavior of iteratively decoded parallel concatenated codes," *IEEE Transactions On Communications*, vol. 49, no. 10, October 2001.

[16] J. G. Proakis, *Digital communications*. McGraw Hill, 1995.

[17] R. A. Askey and A. B. O. Daalhuis, "Generalized hypergeometric function," in *NIST Handbook of Mathematical Functions*. Cambridge University Press, 2010.

[18] R. Gallager, "Low-density parity-check codes," Ph.D. dissertation, MIT, 1963.

[19] F. Lehmann and G. M. Maggio, "Analysis of the iterative decoding of LDPC and product codes using the Gaussian approximation," *IEEE Transactions on Information Theory*, vol. 49, no. 11, pp. 2993–3000, November 2003.