Adaptive Multilevel Coding in OFDM Systems

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***Abstract*— A link adaptation method for OFDM systems based on multilevel coding is presented. A set of multilevel codes to be used for link adaptation is constructed based on capacity-approaching LDPC codes. The performance of the suggested system is analyzed.**

I. INTRODUCTION

Orthogonal frequency division multiplexing (OFDM) is considered the leading candidate for high data rate transmissions of 4-th generation mobile systems, especially in downlink. The main reason for this is the relatively simple receiver architecture also in highly frequency-selective multipath channels. However, it is well known that link adaptation is essential in order to exploit all OFDM benefits. Most of the existing link adaptation algorithms for OFDM deal with adaptive modulation [1], [2], [3]. If coding is used in an adaptive system, coding and modulation should be optimized jointly, i.e. adaptive coded modulation should be used. Most of the existing adaptive coded modulation techniques for OFDM are either extensions of similar single-carrier methods [4] or employ bit interleaved coded modulation [5]. In both cases the performance of an adaptive system depends on the number of different data transmission schemes implemented, i.e. on the granularity of data rate adjustment. Clearly, the complexity of the system increases with the number of different coding/modulation methods supported.

In this paper we present an adaptive system based on multilevel coding (MLC) [6] which allows one to fine-tune the data rate and power allocation for each sub-carrier. Application of MLC allows us to implement many different code rates using relatively small number of component codes and modulation formats reducing thus implementation complexity. Here we consider LDPC [7] component codes since they are known to provide very good capacity-approaching performance with practical implementation complexity. However, we would like to emphasize that most of the derivations presented here are independent of actual component codes used.

The adaptive system described here is based on the well-known waterfilling principle [8]. While this principle is not new, it appears that up to now very little is known about its statistical behavior. Therefore we present also a novel method for statisitcal performance evaluation of waterfilling-based systems.

The remainder of the paper is organized as follows. In Section II we describe our adaptative system. Section III presents its analysis. Section IV presents some simulation results. Finally, conclusions are drawn in Section V.

II. SYSTEM DESCRIPTION

A. Selecting LDPC codes

Since low-density parity check (LDPC) codes are known to be capacity-approaching, it is desired to use them in an adaptive multilevel system. LDPC codes are defined by a sparse parity check matrix, and find a convenient representation in form of Tanner graphs. The interested reader is referred to [9] for all related definitions. Most of the existing LDPC code families can be roughly classified as follows:

- Irregular pseudo-random constructions. These codes can perform very closely to the channel capacity provided that the degree distributions of their Tanner graphs are optimized using density evolution [10]. However, for high code rate their performance is much worse than it is predicted by density evolution due to finite-length effects [11].
- Algebraic constructions. Most of these codes are regular and lack thus the capacity-approaching behavior. However, their minimum distance is typically quite good ensuring thus good performance. On the other hand, most algebraic LDPC codes have very high rate.

Since implementation of multilevel coding requires both low and high rate codes, both irregular pseudo-random and regular algebraic codes have to be used. In this paper we consider Progressive Edge Growth (PEG) [12] and Reed-Solomon based (RSLDPC) [13] LDPC code constructions, respectively. Infinite-length LDPC codes can be characterized by the iterative decoding threshold (which can be computed numerically) [10], so that if the channel quality measure (i.e. SNR) is higher than this threshold value, iterative decoding is always successful. This quantity provides also a good performance measure of finite length codes. Figure 1 presents the curves of asymptotic iterative decoding threshold vs. code rate for the considered code families[1]. One can see that the RSLDPC curve is almost flat for code rate $0.7 \leq r \leq 0.8$. Note that one needs extremely high number of redundant parity checks in order to achieve code rate $r \approx 0.67$. This causes the threshold SNR to decrease considerably in this region. Note

[1]Degree distributions for PEG codes were obtained from the online database at `http://lthcwww.epfl.ch/research/ldpcopt/`. Parameters of RSLDPC codes are described in [13].

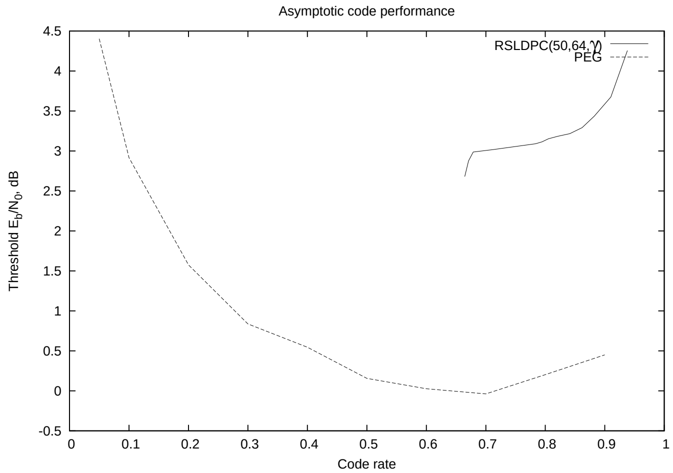

Fig. 1. Asymptotic performance of the considered LDPC codes.

that the parity check matrices of PEG codes typically do not have such redundant parity checks. On the other hand, PEG algorithm fails to produce good high-rate LDPC codes. Hence it is reasonable to use RSLDPC codes with rate not less than 0.8 and PEG codes with lower rates.

B. Constructing a family of multilevel codes

In order to flexibly adjust data transmission rate for each sub-carrier in an OFDM system, a number of different data transmission schemes should be available. Let us now consider a novel pragmatic method for obtaining a number of multilevel codes with high rate granularity.

Multilevel coding [6] represents a practical method for decoupling optimization of coding and modulation parts of the transmitter. Each point of the signal constellation being used is assigned with an unique index j. Assuming that binary codes are used, each index can be represented as an l-tuple $j = (x_0, x_1, \ldots, x_{l-1}), x_i \in \{0, 1\}, i = 0..l-1$. Now the data stream can be partitioned into l substreams. Each substream (*level*) can be encoded using its own error correcting code of length n and symbols of the codewords $c_i = (c_{i0}, \ldots, c_{i,n-1})$ obtained can be used to select n signals $a_{(c_{0,k}, c_{1,k}, \ldots, c_{l-1,k})}, k = 0..n-1$ to be transmitted, where a_j denotes the signal point with index j. Then transmission of symbols with indices $(c_{0,k}, c_{1,k}, \ldots, c_{l-1,k})$ over the physical channel can be separated into the parallel transmission of individual digits c_{ik} over l equivalent channels. Each equivalent channel can be characterized by its capacity C_i.

A number of design rules for multilevel codes are known [6]. In principle, the capacity can be approached by assigning to each level a code with rate equal to the capacity of the corresponding equivalent channel (capacity rule). However, this requires infinitely long capacity approaching codes. More practical equal error probability code design rule states that decoding error probabilities should be the same for all levels. However, their evaluation requires knowledge of code weight enumerators which are in general quite difficult to obtain. Therefore we suggest the following approach for construction of a family of multi-level codes. For the sake of simplicity we describe it initially for the case of M-PAM modulation.

1) Construct a number K of component binary codes with different rates (e.g. $0.05 \leq r_k \leq 0.95, k = 1..K$) according e.g. to the guidelines presented above for LDPC codes.
2) Use theoretical analysis or simulations to obtain functions $p_k(\gamma)$ giving the bit error probability of the k-th code in an AWGN channel with binary signalling and SNR γ. Solve the equations $p_k(\gamma) = p_0$ to obtain the SNR γ_k required by k-th code to achieve the target BER p_0. This gives a mapping $r(\gamma)$ specifying the component code rate r to be used in a channel with SNR γ. On the other hand, this can be also considered as a mapping $r(C(\gamma))$ specifying the code rate to be used in a channel with capacity $C(\gamma)$.
3) Consider a number of different 2^l-PAM signal constellations. For each $l = 1..L$ consider different values of SNR $\gamma = \gamma_0 \Delta^j, j = 0, 1, \ldots$, where γ_0 and Δ are some positive constants. Here it is assumed that the signal power is normalized to 1. For each γ evaluate equivalent sub-channel capacities $C_i, i = 0..l-1$ and select for each level a component code with rate $r(C_i)$, rounding it down if necessary. This step is based on the assumption that the performance of component codes in an equivalent sub-channel of a multi-level code does not deviate significantly from the one for the case of AWGN channel. For each γ select the 2^l-PAM constellation providing the greatest total multilevel code rate $R = \sum_{i=0}^{l-1} \lfloor r(C_i) \rfloor$, where $\lfloor \cdot \rfloor$ denotes rounding down to the closest available code rate.

This method can be used to obtain a list of multilevel codes with different rates R suitable for data transmission over AWGN channel with different values of SNR γ. By considering M-QAM square constellation as a superposition of two $\sqrt{M}$-PAM constellations one can easily obtain QAM-based multilevel codes. It was found (for QAM-based codes) that the function $R(\gamma)$ specifying the multilevel code rate to be used for the case of SNR equal to γ can be computed with very good accuracy using well-known ”gap approximation” [14]

$$R(\gamma) \approx \log_2(1 + \frac{\gamma}{\Gamma}), \tag{1}$$

where Γ denotes gap to capacity, which depends on the properties of the component codes used. This approach allows one to obtain a great number of multilevel codes with different rates using relatively small number of component binary codes.

C. Adaptive rate allocation and power loading

Let us consider an OFDM system with N sub-carriers. The received signal r_i for each sub-carrier can be represented in frequency domain as

$$r_i = \mu_i s_i + \eta_i, i = 1..N \tag{2}$$

where s_i is the signal being transmitted over i-th sub-carrier, μ_i is the complex-valued channel transfer factor and η_i is the sample of Gaussian noise with variance σ^2. The system performance depends on the sub-carrier SNR $\gamma_i = V_i^2\xi_i$, where $\xi_i = \frac{|\mu_i|^2}{\sigma^2}$ is the sub-carrier channel-to-noise ratio and $V_i^2 = E[|s_i|^2]$ is the sub-carrier power gain. The most straightforward way to implement adaptive data transmission is to assign to each sub-carrier its own coding scheme with some data rate R_i. Consider the problem of finding the power and rate allocation $(V_i^2, R_i), i = 1..N$ such that the total data rate is equal to some target value R_0 and the total transmitter power $\sum_{i=1}^N V_i^2$ is minimized. This is a classical constrained optimization problem with Lagrange function

$$\mathcal{L}(V_i, \lambda) = \sum_{i=1}^N V_i^2 - \lambda \left(\sum_{i=1}^N \log_2 \left(1 + \frac{V_i^2 \xi_i}{\Gamma} \right) - R_0 \right) \quad (3)$$

After some derivations this leads to the following system of equations:

$$V_i^2 = \max\left(0, \frac{\lambda}{\ln 2} - \frac{\Gamma}{\xi_i}\right), i = 1..N \quad (4)$$

$$R_0 = \sum_{i=1}^N \log_2 \left(1 + \frac{V_i^2 \xi_i}{\Gamma} \right) \quad (5)$$

This system can be efficiently solved using bisection on λ (similarly to [15]) giving transmitter gains V_i for each sub-carrier. Code rates R_i for each sub-carrier can be easily computed using (1). Observe that the bit error rate requirements are automatically taken into account by gap approximation. Moreover, the actual optimization problem requires that the code rates assigned to sub-carriers must belong to the list of available multilevel codes, that is they must be discrete, while equations (4)– (5) assume that they are continuous. Therefore, good quality solution can be obtained provided that sufficiently many data transmission schemes with different code rates are available.

However, performing coding independently for each sub-carrier cannot be considered as a practical solution due to extremely high latency and complexity. It was also observed that code allocations produced by the above algorithm typically employ only a few different multilevel codes. Therefore it is natural to distribute the symbols of the same multilevel codeword over a number of sub-carriers. Since sub-carriers with close channel-to-noise ratios ξ_i are likely to be assigned with the same multi-level code, it is reasonable to perform optimization not over sub-carriers, but over sub-bands. Sub-bands can be formed by sorting the sub-carriers according to their CNRs ξ_i and grouping close ones into sub-bands of size L. The sub-band CNR can be defined as a geometric average [16] of sub-carrier CNRs. This approach seems to be more advantageous compared to grouping adjacent sub-carriers into sub-bands [4], since the latter one effectively reduces the channel frequency selectivity.

III. Performance of Adaptive Systems Based on Waterfilling

System of equations (4)–(5) represent the classical waterfilling power distribution. Its solution can be easily obtained numerically for each *particular* set of CNRs ξ_i. However, in many cases these variables are random and one needs to evaluate the average system performance. This kind of analysis may be needed not only in the design of single-user adaptive systems, but also in multi-user systems, where one needs to estimate the number of sub-carriers to be assigned to each user. For example, the adaptive algorithm presented in [17] requires this in order to adaptively allocate sub-carriers to users in an OFDM system. The main difficulty here is caused by the $\max(\cdot)$ function in (4), i.e. by the existence of unused sub-carriers. Almost all existing work on this issue (cf. [18], [19]) is based either on the assumption that all sub-carriers are used (i.e. transmitter power is sufficiently high) or on simulations. The former assumption is not always valid, while the latter approach is in general extremely time-consuming.

Let us assume that the channel transfer factors for individual sub-carriers are independent and identically distributed. Re-order sub-carriers according to their CNR. Then the system of equations (4)–(5) can be rewritten using $\xi_{i:N}$ instead ξ_i, where $\xi_{i:N}$ is the i-th order statistic of the array $(\xi_1, \ldots, \xi_N)$. Clearly, this does not affect system performance. If N is sufficiently large, it is possible to show [20] that almost all values $\xi_{i:N}$ have normal distribution with mean value $M[\xi_{i:N}] = F^{-1}(p)$ and variance $D[\xi_{i:N}] = \frac{p(1-p)}{nf^2(F^{-1}(p))}$, where $p = \frac{i}{N}$, $F(x)$ is the cumulative distribution function of ξ and $f(x)$ is the corresponding probability density function. If the channel experiences independent Rayleigh fading, the expressions for mean value and variance can be transformed to

$$M[\xi_{i:N}] \approx \ln \frac{N}{N-i} \quad (6)$$

and

$$D[\xi_{i:N}] \approx \frac{(i/N)(1-i/N)}{N \exp(-2 \ln \frac{N}{N-i})} = \frac{i}{N(N-i)} \quad (7)$$

One can see that for sufficiently large N and $i < N$ the variance of $\xi_{i:N}$ is quite small. Hence it is possible to neglect randomness of $\xi_{i:N}$ and consider them as deterministic values. However, these expressions are valid only for $i < N - \delta$ and sufficiently large δ. For $i = N$ (i.e. the best sub-carrier) it is possible to show that $M[\xi_{N:N}] \approx \ln(N) - 0.5772$ and $D[\xi_{N:N}] \approx \frac{\pi^2}{6} \approx 1.64$. It is known also that $\xi_{N:N} - \ln(N)$ has Gumbel distribution, i.e.

$$P\{\xi_{N:N} - \ln(N) \leq x\} \approx \exp(-\exp(-x))$$

Observe also that the expression (6) leads to $M[\xi_{N-1:N}] = \ln N$, which is slightly more than the exact value of $M[\xi_{N:N}]$, and $M[\xi_{N:N}] = \infty$, which does not make sense. However, for sufficiently large N these effects can be avoided by excluding $\xi_{N:N}$ from further consideration.

Using $M[\xi_{i:N}]$ instead of $\xi_{i:N}$ one can solve the system of equations (4)–(5) numerically and obtain average system performance without running time-consuming simulations. Moreover, this enables one to estimate the number of sub-carriers actually used in the low-power region. Indeed, to do this one needs to find the smallest i s.t. $\frac{\lambda}{\ln 2} - \frac{\Gamma}{\xi_{i:N}} > 0$, and $\xi_{i:N}$ can be again replaced with its mean value. Then the

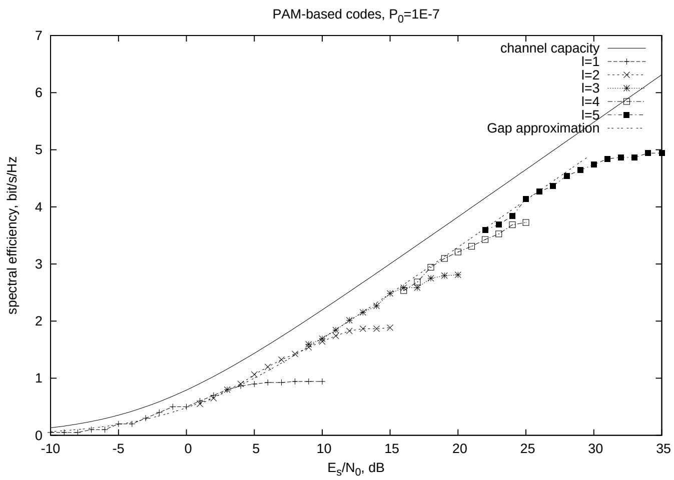

Fig. 2. Spectral efficiency of PAM-based multilevel codes.

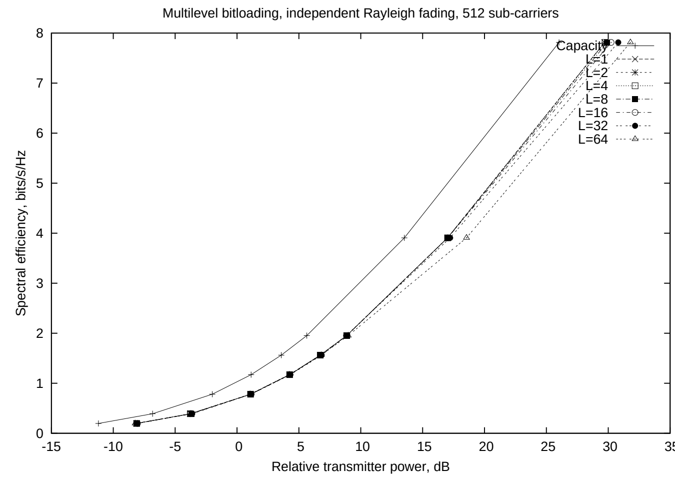

Fig. 3. Spectral efficiency of an adaptive OFDM system.

number of actually used sub-carriers is equal to $N - i + 1$. This approach may be useful in multi-user systems where only a small fraction of sub-carriers may be utilized by each user. Observe also, that by setting $\Gamma = 1$ these derivations can be used for capacity calculations.

IV. Numerical Results

In this section we present the numerical results illustrating the performance of the suggested adaptive system as well as the validity of the above derivations. The results are reported in terms of the system spectral efficiency and relative transmitter power. Relative transmitter power is defined as $\frac{\sum_{i=1}^{N} V_i^2}{N_0}$, where N_0 is the AWGN power spectrum density.

Figure 2 presents the curves illustrating the spectral efficiency of the obtained set of PAM-based multilevel codes. Here 17 component LDPC codes of length 3200 were used to obtain 98 different multi-level codes. One can see that the constructed set of codes performs within 3 dB to Shannon limit at BER 10^{-7}. As it can be seen in Figure 3, this 3 dB gap is preserved in an adaptive system operating over the independent Rayleigh fading channel. Moreover, it can be seen that optimization over sub-bands of size $L > 1$ results in very small loss in spectral efficiency. Figure 4 presents the comparison of results obtained via the order statistics method presented in Section III and by means of simulations. One can see that the capacity curves obtained by both methods are almost identical, while the curve corresponding to the simulated performance of the adaptive system is slightly worse in the high-rate region than its theoretical estimate. The reason for this is that the adaptive system must employ a finite set of data transmission schemes (i.e. multi-level codes) and some sub-carriers in this region are forced to use the best multi-level code available, i.e. the system performance could be improved by constructing more multi-level codes.

Figure 5 presents the simulation results illustrating the relative transmitter power needed to achieve data rate (or capacity) $R_0 = 2000$ bits/OFDM symbol, i.e. spectral efficiency 3.9

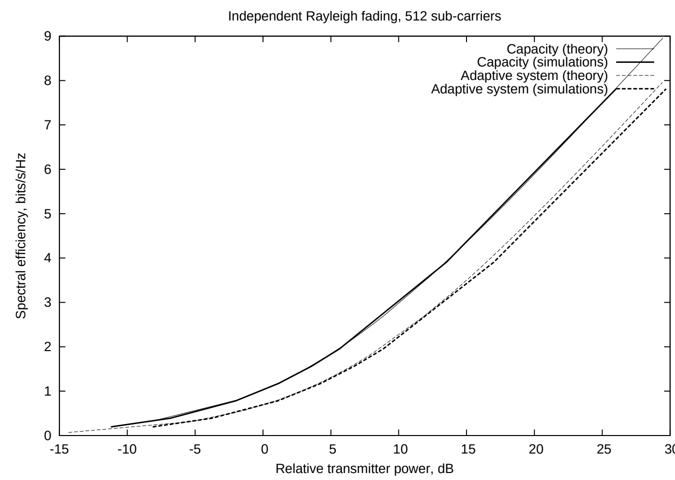

Fig. 4. Comparison of theoretical and simulation-based performance assessment methods

bits/s/Hz, for the case of WSSUS channel with exponential power delay profile [21] for different values of maximal delay spread τ_{max} corresponding to the 30 dB attenuation point. One can see that even for moderate τ_{max} the adaptive system performs quite close to the lower bound given by the performance of the system operating over the independent Rayleigh fading channel. However, for very small values of τ_{max} (i.e. strong correlation of channel transfer factors) the deviation becomes significant.

V. Conclusions

In this paper we presented an adaptive OFDM system with optimized coded modulation and power allocation. Application of multilevel coding enables one to improve the flexibility of the adaptive system since it allows one to obtain many different data transmission schemes (i.e. multi-level codes) using relatively small amount of component codes. It was shown that by grouping sub-carriers with close CNRs into sub-bands and performing optimization over them the system

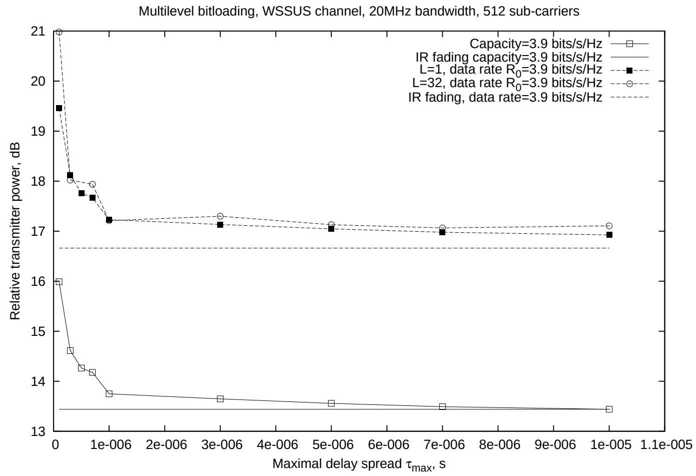

Fig. 5. Comparison of the theoretical and simulation-based performance assessment methods

complexity can be considerably reduced with very small performance loss. The described approach can be also used in adaptive multi-user OFDM systems.

We have also shown that the performance of waterfilling-based adaptive systems can be efficiently assessed by considering the order statistics of the channel-to-noise ratios. This allows also one to estimate the number of sub-carriers needed by a user to support some target data rate.

At the time of preparation of the final version of this paper similar adaptive system has been presented in [22]. It is shown there that similar performance can be achieved by means of combined bit-interleaved coded modulation and Reed-Solomon coding. However, application of very long LDPC codes (length 100000) is suggested.

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